14.7, F20-e3-4: **Find the local maximum points, local minimum points, and saddle points of $f(x,y) = x^2y + 4xy - 2y^2$.**

Answer:

The derivatives are

$$f_x = 2xy + 4y, \quad f_y = x^2 + 4x - 4y,$$

$$f_{xx} = 2y, \quad f_{xy} = 2x + 4, \quad \text{and} \quad f_{yy} = -4.$$

Observe that the equation $f_x = 0$ can be factored to yield $2y(x+2) = 0$. There are two different ways this equation can be satisfied: either $y = 0$ or $x = -2$.

When $y = 0$, then $f_y = 0$ becomes $x^2 + 4x = 0$; hence $x(x+4) = 0$ and $x = 0$ or $x = -4$. So far, we have identified two critical points; namely $(0,0)$ and $(-4,0)$.

When $x = -2$, then $f_y = 0$ becomes $4 - 8 - 4y = 0$; hence, $-4 = 4y$ and $y = -1$.

Thus f has exactly three critical points, namely $(0,0)$, $(-4,0)$, and $(-2,-1)$.

We apply the second derivative test at each critical point.

At $(0,0)$, the Hessian $H|_{(0,0)}$ is equal to

$$H|_{(0,0)} = (f_{xx}f_{yy} - f_{xy}^2)|_{(0,0)} = \Big((2y)(-4) - (2x+4)^2\Big)\Big|_{(0,0)} = -16 < 0.$$

Thus,

$$\boxed{(0,0,f(0,0)) \text{ is a saddle point.}}$$

At $(-4,0)$, the Hessian $H|_{(-4,0)}$ is equal to

$$H|_{(-4,0)} = (f_{xx}f_{yy} - f_{xy}^2)|_{(-4,0)} = \Big((2y)(-4) - (2x+4)^2\Big)\Big|_{(-4,0)} = -16 < 0.$$

Thus,

$$\boxed{(-4,0,f(-4,0)) \text{ is a saddle point.}}$$

At $(-2,-1)$, the Hessian $H|_{(-2,-1)}$ is equal to

$$H|_{(-2,-1)} = (f_{xx}f_{yy} - f_{xy}^2)|_{(-2,-1)} = \Big((2y)(-4) - (2x+4)^2\Big)\Big|_{(-2,-1)}$$

$$= \Big((-2)(-4) - (2(-2)+4)^2\Big)\Big|_{(-4,0)} = 8 > 0.$$

We see also that $f_{xx}(-2,-1) = 2y|_{(-2,-1)} = -2 < 0$. We conclude that

$$\boxed{(-2,-1,f(-2,-1)) \text{ is a local maximum.}}$$