14.7, number 13: **Find all local maxima, local minima, and saddle points of $f(x,y) = x^3 - y^3 - 2xy + 6$.**

**Answer:** We compute

$$\begin{aligned} f_x =& 3x^2 - 2y \\ f_y =& -3y^2 - 2x \\ f_{xx} =& 6x \\ f_{xy} =& -2 \\ f_{yy} =& -6y \end{aligned}$$

We see that $f_x$ and $f_y$ are zero when both equations:

$$\begin{cases} 3x^2 - 2y = 0 \\ -3y^2 - 2x = 0 \end{cases}$$

are satisfied. Re-write the equations as

$$\begin{cases} \frac{3x^2}{2} = y \\ -3y^2 - 2x = 0 \end{cases}$$

and

$$\begin{cases} \frac{3x^2}{2} = y \\ -3(\frac{3x^2}{2})^2 - 2x = 0 \end{cases} \tag{3}$$

The bottom equation from (3) is

$$\frac{-27}{4}x^4 - 2x = 0$$

$$27x^4 + 8x = 0$$

$$x(27x^3 + 8) = 0$$

So $x = 0$ or $x^3 = \frac{-8}{27}$. The number $\frac{-8}{27}$ has exactly one cube root; namely, $\frac{-2}{3}$. The system of equations (3) has two solutions; namely $(0,0)$ and $(-\frac{2}{3}, \frac{2}{3})$. We must deal with each of these critical points.

We apply the Second Derivative Test at $(0,0)$:

$$(f_{xx}f_{yy} - {f_{xy}}^2)|_{(0,0)} = 0 \cdot 0 - (-2)^2,$$

which is negative; hence $(0,0,f(0,0))$ is a saddle point on the graph of $z = f(x,y)$.

We apply the Second Derivative Test at $(-\frac{2}{3}, \frac{2}{3})$:

$$(f_{xx}f_{yy} - {f_{xy}}^2)|_{(-\frac{2}{3},\frac{2}{3})} = 6(-\tfrac{2}{3})(-6)(\tfrac{2}{3}) - (-2)^2 = (-4)(-4) - 4 = 12,$$

which is positive. Thus $f$ has a local maximum or a local minimum at this point. Observe that $f_{xx}|_{(-\frac{2}{3},\frac{2}{3})}$ is negative. We conclude that $(-\frac{2}{3}, \frac{2}{3}, f((-\frac{2}{3}, \frac{2}{3}))$ is a local maximum point of the function $f$. We conclude that

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| $(0, 0, f(0, 0))$ is a saddle point on the graph of $z = f(x, y)$ and<br>$(-\frac{2}{3}, \frac{2}{3}, f((-\frac{2}{3}, \frac{2}{3}))$ is a local maximum point on the graph of $z = f(x, y)$. |