14.4, number 7: **Consider $z = 4e^x \ln y$, $x = \ln(u\cos v)$, and $y = u\sin v$.**

(a) **Use the chain rule to calculate $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$. Express your answers in terms of $u$ and $v$. (In other words, there should be no $x$'s and $y$'s in your answer.)**

(b) **First write $z$ as a function of $u$ and $v$ directly, then compute $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$.**

(c) **Evaluate $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ at $(u, v) = (2, \frac{\pi}{4})$.**

**Answer:**

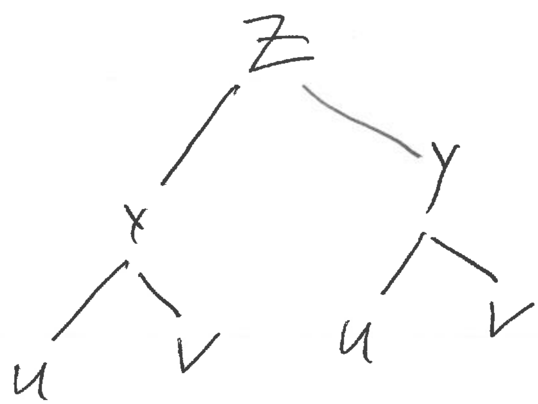
(a) Observe that $\frac{\partial z}{\partial x} = 4e^x \ln y$, $\frac{\partial x}{\partial u} = \frac{1}{u}$, $\frac{\partial z}{\partial y} = \frac{4e^x}{y}$, and $\frac{\partial y}{\partial u} = \sin v$. We drew a tree on the next page.

$$\begin{aligned}
\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial x}{\partial u} \\
&= 4e^x \ln y \frac{1}{u} + \frac{4e^x}{y}\sin v \\
&= \frac{4e^{\ln(u\cos v)}\ln(u\sin v)}{u} + \frac{4e^{\ln(u\cos v)}}{u\sin v}\sin v \\
&= \frac{4(u\cos v)\ln(u\sin v)}{u} + \frac{4u\cos v}{u\sin v}\sin v \\
&= \boxed{4\cos v\ln(u\sin v) + 4\cos v}
\end{aligned}$$

In a similar manner, we see that $\frac{\partial z}{\partial x} = 4e^x \ln y$, $\frac{\partial x}{\partial v} = \frac{-\sin v}{\cos v}$, $\frac{\partial z}{\partial y} = \frac{4e^x}{y}$, and $\frac{\partial y}{\partial v} = u\cos v$. Use the tree on the next page to see that

$$\begin{aligned}
\frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} \\
&= 4e^x \ln y \frac{(-\sin v)}{\cos v} + \frac{4e^x}{y} u\cos v \\
&= 4e^{\ln(u\cos v)}\ln(u\sin v)\frac{(-\sin v)}{\cos v} + \frac{4e^{\ln(u\cos v)}}{u\sin v} u\cos v \\
&= -4u\cos v\frac{\sin v}{\cos v}\ln(u\sin v) + \frac{4u(\cos^2 v)u}{u\sin v} \\
&= \boxed{-4u\sin v\ln(u\sin v) + \frac{4u\cos^2 v}{\sin v}}
\end{aligned}$$

# Picture for 14.4 Number 7



(b) We compute

$$\begin{aligned} z &= 4e^x \ln y \\ &= 4e^{\ln(u\cos v)} \ln(u \sin v) \\ &= 4u\cos v \ln(u\sin v); \end{aligned}$$

hence

$$\begin{aligned} \frac{\partial z}{\partial u} &= 4u\cos v \cdot \frac{1}{u} + [\ln(u\sin v)] \cdot 4\cos v \\ \frac{\partial z}{\partial u} &= \boxed{4\cos v + [\ln(u\sin v)] \cdot 4\cos v} \end{aligned}$$

and

$$\begin{aligned} \frac{\partial z}{\partial v} &= 4u\cos v \frac{\cos v}{\sin v} - 4u\sin v\ln(u\sin v) \\ &= \boxed{\frac{4u\cos^2 v}{\sin v} - 4u\sin v \ln(u\sin v).} \end{aligned}$$

Observe that (a) and (b) gave the same answer.

(c) We see that

$$\begin{aligned} \left.\frac{\partial z}{\partial u}\right|_{(u,v)=(2,\frac{\pi}{4})} &= \Big(4\cos v + [\ln(u\sin v)] \cdot 4\cos v\Big)\Big|_{(u,v)=(2,\frac{\pi}{4})} \\ &= 4\cos\tfrac{\pi}{4} + [\ln(2\sin\tfrac{\pi}{4})] \cdot 4\cos\tfrac{\pi}{4} \\ &= 4\tfrac{\sqrt{2}}{2} + [\ln(2(\tfrac{\sqrt{2}}{2}))] \cdot 4(\tfrac{\sqrt{2}}{2}) \\ &= 2\sqrt{2} + \ln(\sqrt{2}) \cdot 2\sqrt{2} && \text{This much is fine.} \\ &= 2\sqrt{2} + \tfrac{1}{2}(\ln 2) 2\sqrt{2} \\ &= \boxed{2\sqrt{2} + (\ln 2)\sqrt{2}} \\ \left.\frac{\partial z}{\partial v}\right|_{(u,v)=(2,\frac{\pi}{4})} &= \Big(\frac{4u\cos^2 v}{\sin v} - 4u\sin v\ln(u\sin v)\Big)\Big|_{(u,v)=(2,\frac{\pi}{4})} \\ &= \tfrac{4(2)(\cos^2\frac{\pi}{4})}{\sin(\frac{\pi}{4})} - 4(2)\sin(\tfrac{\pi}{4})\ln(2\sin(\tfrac{\pi}{4})) \\ &= \tfrac{8(\frac{\sqrt{2}}{2})^2}{\frac{\sqrt{2}}{2}} - 8(\tfrac{\sqrt{2}}{2})\ln(2(\tfrac{\sqrt{2}}{2})) \\ &= 8(\tfrac{\sqrt{2}}{2}) - 8(\tfrac{\sqrt{2}}{2})\ln(\sqrt{2}) \\ &= 4\sqrt{2}(1 - \ln(\sqrt{2})) && \text{This much is fine.} \\ &= 4\sqrt{2}(1 - \tfrac{1}{2}\ln(2)) \\ &= \boxed{2\sqrt{2}(2 - \ln(2))} \end{aligned}$$