13.3, number 15: **Find the length of the curve**

$$\overrightarrow{\boldsymbol{r}}(t) = (\sqrt{2}t)\overrightarrow{\boldsymbol{i}} + (\sqrt{2}t)\overrightarrow{\boldsymbol{j}} + (1-t^2)\overrightarrow{\boldsymbol{k}}$$

from $(0,0,1)$ to $(\sqrt{2},\sqrt{2},0)$.

Answer: Observe that an object with position vector $\overrightarrow{\boldsymbol{r}}(t)$ is standing on $(0,0,1)$ at time $t=0$ and is standing on $(\sqrt{2},\sqrt{2},0)$ at time $t=1$. The length of the curve is

$$\begin{aligned}\int_0^1 |\overrightarrow{\boldsymbol{r}}'(t)|dt &= \int_0^1 |\sqrt{2}\,\overrightarrow{\boldsymbol{i}} + \sqrt{2}\,\overrightarrow{\boldsymbol{j}} - 2t\,\overrightarrow{\boldsymbol{k}}|dt\\ &= \int_0^1 \sqrt{2+2+4t^2}dt\\ &= 2\int_0^1 \sqrt{1+t^2}dt\end{aligned}$$

Look at Formula 35 on page T-2 at the back of the book, or use a search engine, or use Trig substitution to calculate $\int \sqrt{1+t^2}dt = \frac{t}{2}\sqrt{1+t^2} + \frac{1}{2}\ln(t+\sqrt{1+t^2}) + K$. I will demonstrate the Trig substitution at the end of this solution.

$$\begin{aligned}&= 2\Big[\frac{t\sqrt{1+t^2}}{2} + \frac{1}{2}\ln(t+\sqrt{1+t^2})\Big]_0^1\\ &= 2\Big[\frac{\sqrt{2}}{2} + \frac{1}{2}\ln(1+\sqrt{1+1^2}) - \Big(\frac{0}{2} + \frac{1}{2}\ln(1)\Big)\Big]\\ &= \boxed{\sqrt{2} + \ln(1+\sqrt{2}).}\end{aligned}$$

To calculate $\int \sqrt{1+t^2}dt$, let

$$t = \tan\theta. \qquad (1)$$

Recall that

$$1 + \tan^2\theta = \sec^2\theta.$$

(This is merely $\sin^2\theta + \cos^2\theta = 1$ divided by $\cos^2\theta$.) So,

$$\sqrt{1+t^2} = \sec\theta. \qquad (2)$$

Take the derivative of $t = \tan\theta$ with respect to θ to learn that

$$dt = \sec^2\theta d\theta.$$

We now see that

$$\int \sqrt{1+t^2}dt = \int \sec^3\theta d\theta.$$

Use Integration by Parts to compute $\int \sec^3\theta d\theta$. Recall that Integration by Parts says that

$$\int u\,dv = uv - \int v\,du.$$

We take

$$u = \sec\theta \quad \text{and} \quad dv = \sec^2\theta d\theta$$

(because it is easy to integrate $\sec^2\theta$). We compute

$$du = \sec\theta\tan\theta d\theta \quad \text{and} \quad v = \tan\theta.$$

We now have

$$\begin{aligned}
\int \sec^3\theta\, d\theta = \int u\, dv = uv - \int v du \\
&= \sec\theta\tan\theta - \int \sec\theta\tan^2\theta\, d\theta \\
&= \sec\theta\tan\theta - \int \sec\theta(\sec^2\theta - 1)\, d\theta \\
&= \sec\theta\tan\theta - \int \sec^3\theta\, d\theta + \int \sec\theta\, d\theta.
\end{aligned}$$

Add $\int \sec^3\theta\, d\theta$ to both sides to learn that

$$2\int \sec^3\theta\, d\theta = \sec\theta\tan\theta + \int \sec\theta\, d\theta.$$

In other words

$$\int \sec^3\theta\, d\theta = \frac{1}{2}\Big[\sec\theta\tan\theta + \int \sec\theta\, d\theta\Big]. \tag{3}$$

We still must compute $\int \sec\theta\, d\theta$. The trick for integrating $\sec\theta$ is to multiply the top and bottom by

$$\sec\theta + \tan\theta.$$

So

$$\int \sec\theta\, d\theta = \int \sec\theta \left(\frac{\sec\theta + \tan\theta}{\sec\theta + \tan\theta}\right) d\theta.$$

Notice that the derivative of the denominator $\sec\theta + \tan\theta$ is exactly the numerator, namely $\sec\theta\tan\theta + \sec^2\theta$. It follows that

$$\int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C.$$

We insert the above calculation into equation (3).

$$\int \sec^3\theta\, d\theta = \frac{1}{2}\Big[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta| + C\Big]$$

Let K represent the constant $\frac{C}{2}$. We have calculated that

$$\int \sqrt{1+t^2}\, dt = \int \sec^3\theta\, d\theta = \frac{1}{2}\Big[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\Big] + K.$$

Use equations (1) and (2) to conclude that

$$\int \sqrt{1+t^2}\, dt = \frac{1}{2}\Big[t\sqrt{1+t^2} + \ln|t + \sqrt{1+t^2}|\Big] + K.$$