13.2, number 24:

a. **Show that doubling a projectile's initial speed at a given launch angle multiplies its range by a factor of** 4**.**

b. **By about what percentage should you increase the initial speed to double the height and the range?**

Answer: Let $\vec{\boldsymbol{r}}(t) = x(t)\vec{\boldsymbol{i}} + y(t)\vec{\boldsymbol{j}}$ be the position vector of the object at time t.

We know that

$\vec{\boldsymbol{r}}''(t) = -g\vec{\boldsymbol{j}}$	Gravity is the only force acting on the projectile.
$\vec{\boldsymbol{r}}'(0) = v_0(\cos\alpha\,\vec{\boldsymbol{i}} + \sin\alpha\,\vec{\boldsymbol{j}})$	See the final page of the solution, if necessary.
$\vec{\boldsymbol{r}}(0) = 0$	We launch the projectile from the origin.

Integrate $\vec{\boldsymbol{r}}''$ and use the value of $\vec{\boldsymbol{r}}'(0)$ to evaluate the constant in order to find

$$\vec{\boldsymbol{r}}'(t) = v_0\cos\alpha\,\vec{\boldsymbol{i}} + (v_0\sin\alpha - gt)\vec{\boldsymbol{j}}.$$

Integrate $\vec{\boldsymbol{r}}'(t)$ and use the value of $\vec{\boldsymbol{r}}(0)$ to evaluate the constant in order to find

$$\vec{\boldsymbol{r}}(t) = (v_0\cos\alpha)t\,\vec{\boldsymbol{i}} + \big((v_0\sin\alpha)t - \frac{g}{2}t^2\big)\vec{\boldsymbol{j}}.$$

Of course, the projectile is on the ground when the $\vec{\boldsymbol{j}}$-component of $\vec{\boldsymbol{r}}(t)$ is zero. That, is the projectile is on the ground when

$$(v_0\sin\alpha)t - \frac{g}{2}t^2 = 0$$

$$t(v_0\sin\alpha - \frac{g}{2}t) = 0$$

So, $t = 0$ or $t = \frac{2}{g}v_0\sin\alpha$. The time $t = 0$ is the beginning of the projectile's flight. The time $t = \frac{2}{g}v_0\sin\alpha$ is the end of the flight. The x-coordinate at the end of the flight is the range of the projectile. So, the range of the projectile is

$$(v_0\cos\alpha)\frac{2}{g}v_0\sin\alpha = \frac{2v_0^2\cos\alpha\sin\alpha}{g}.$$

> If everything is kept the same, except v_0 is replaced by $2v_0$, then the range of the projectile is $\frac{2(2v_0)^2\cos\alpha\sin\alpha}{g} = 4(\frac{2v_0^2\cos\alpha\sin\alpha}{g})$, which is 4 times the range of the first projectile.

This completes (a).

Now we are ready for (b). We already know from (a) that if we want to change the initial speed in order to double the range of the projectile we should multiply the initial speed by a factor of $\sqrt{2}$. We better check how this change affects the maximum height.

We think about the original set up:

$$\overrightarrow{\boldsymbol{r}}'(t) = v_0 \cos\alpha \,\overrightarrow{\boldsymbol{i}} + (v_0 \sin\alpha - gt)\,\overrightarrow{\boldsymbol{j}}$$
$$\overrightarrow{\boldsymbol{r}}(t) = (v_0 \cos\alpha)t\,\overrightarrow{\boldsymbol{i}} + \big((v_0 \sin\alpha)t - \frac{g}{2}t^2\big)\,\overrightarrow{\boldsymbol{j}}.$$

The maximum height occurs when the velocity in the y-direction is zero. (Or, if you prefer, when $\frac{dy}{dt} = 0$.) This happens when

$$v_0 \sin\alpha - gt = 0$$
$$\frac{v_0 \sin\alpha}{g} = t.$$

The height at that time is

$$y\left(\frac{v_0 \sin\alpha}{g}\right) = (v_0 \sin\alpha)\left(\frac{v_0 \sin\alpha}{g}\right) - \frac{g}{2}\left(\frac{v_0 \sin\alpha}{g}\right)^2 = \frac{v_0^2 \sin^2\alpha}{2g}$$

Once again, we see that if v_0 is replaced by $\sqrt{2}v_0$, then the maximum height is doubled. We conclude that

> if v_0 is replaced by $\sqrt{2}v_0$, then the maximum height and the range of the projectile are both doubled.

Picture for 13.2 Number 24

$$\cos\alpha = \frac{ADJ}{|\vec{r}'(0)|} \qquad \sin\alpha = \frac{OP}{|\vec{r}'(0)|}$$

$$\vec{r}'(0) = ADJ\,\vec{i} + OP\,\vec{j} = |\vec{r}'(0)|\cos\alpha\,\vec{i} + |\vec{r}'(0)|\sin\alpha\,\vec{j}$$
$$= v_0\cos\alpha\,\vec{i} + v_0\sin\alpha\,\vec{j}$$