This problem comes from Fall 2024, Exam 1, number 2.

Find an equation for the plane through the points $P_1 = (1,1,1)$, $P_2 = (-1,2,1)$, and $P_3 = (-2,1,2)$. Check your answer. Make sure it is correct.

Answer: The vector $\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}$ is perpendicular to the plane. We compute

$$\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = \begin{vmatrix} \vec{\boldsymbol{i}} & \vec{\boldsymbol{j}} & \vec{\boldsymbol{k}} \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \vec{\boldsymbol{i}} - \begin{vmatrix} -2 & 0 \\ -3 & 1 \end{vmatrix} \vec{\boldsymbol{j}} + \begin{vmatrix} -2 & 1 \\ -3 & 0 \end{vmatrix} \vec{\boldsymbol{k}}$$
$$= \vec{\boldsymbol{i}} + 2\vec{\boldsymbol{j}} + 3\vec{\boldsymbol{k}}.$$

The point (x,y,z) is on the plane precisely when $\overrightarrow{P_1(x,y,z)}$ is perpendicular to $\vec{\boldsymbol{i}} + 2\vec{\boldsymbol{j}} + 3\vec{\boldsymbol{k}}$.

The point (x,y,z) is on the plane precisely when

$$\overrightarrow{(1,1,1)(x,y,z)} \cdot (\vec{\boldsymbol{i}} + 2\vec{\boldsymbol{j}} + 3\vec{\boldsymbol{k}}) = 0.$$

The equation of the plane is

$$(x-1) + 2(y-1) + 3(z-1) = 0$$

$$\boxed{x + 2y + 3z = 6}$$

Check.

We verify that P_1 satisfies the proposed answer:

$$1(1) + 2(1) + 3(1) = 6.\checkmark$$

We verify that P_2 satisfies the proposed answer:

$$1(-1) + 2(2) + 3(1) = 6.\checkmark$$

We verify that P_3 satisfies the proposed answer:

$$1(-2) + 2(1) + 3(2) = 6.\checkmark$$