12.5, number 29: **Find the equation of the plane which contains the two intersecting lines**

$$L_1 = \begin{cases} x = -1 + t \\ y = 2 + t \\ z = 1 - t \end{cases} \qquad \textbf{and} \qquad L_2 = \begin{cases} x = 1 - 4s \\ y = 1 + 2s \\ z = 2 - 2s. \end{cases}$$

Answer: I suppose that one could trust that the two lines do intersect. I personally would be more comfortable if I **knew** that they intersect. I'll do that first. I'll look for a t and an s so that the x-coordinate of L_1 at time t equals the x-coordinate of L_2 at time s; the y-coordinate of L_1 at time t equals the y-coordinate of L_2 at time s; and the z-coordinate of L_1 at time t equals the z-coordinate of L_2 at time s. In other words, I would solve

$$\begin{cases} -1 + t = 1 - 4s \\ 2 + t = 1 + 2s \\ 1 - t = 2 - 2s \end{cases}$$

simultaneously. The top equation says that $t = 2 - 4s$. I will replace t in equations 2 and 3 with $2 - 4s$. We want to solve

$$\begin{cases} t = 2 - 4s \\ 2 + (2 - 4s) = 1 + 2s \\ 1 - (2 - 4s) = 2 - 2s \end{cases}$$

simultaneously. We want to solve

$$\begin{cases} t = 2 - 4s \\ 3 = 6s \\ -3 = -6s \end{cases}$$

simultaneously. So the position of L_1 at time $t = 0$ is exactly the same as the position of L_2 at time $s = \frac{1}{2}$. Indeed L_1 at $t = 0$ is $(-1, 2, 1)$ and the position of L_2 at $s = \frac{1}{2}$ is $(-1, 2, 1)$. If you skipped the above calculation, but did everything else correctly you would have the right answer. However you would have the right answer only because the people that made the problem; made the problem correctly. However if they had goofed and the two lines did not intersect then there would be no plane that contains both lines. (It is obvious that the two lines are not parallel.) In this case, you would get an answer, but the answer would be meaningless.

Now that we are convinced that the problem can be done, lets do it. Let P_0 be any point on either L_1 or L_2. Let $\vec{v}_1$ be a vector parallel to L_1 and $\vec{v}_2$ be a vector parallel to L_2. We want the equation of the plane through P_0 perpendicular to $\vec{v}_1 \times \vec{v}_2$. I will take P_0 to be the point on L_2 at $s = 0$. This is $P_0 = (1,1,2)$. Of course, $\overrightarrow{\boldsymbol{v}}_1 = \overrightarrow{\boldsymbol{i}} + \overrightarrow{\boldsymbol{j}} - \overrightarrow{\boldsymbol{k}}$ and $\overrightarrow{\boldsymbol{v}}_2 = -4\overrightarrow{\boldsymbol{i}} + 2\overrightarrow{\boldsymbol{j}} - 2\overrightarrow{\boldsymbol{k}}$. We compute $\overrightarrow{\boldsymbol{v}}_1 \times \overrightarrow{\boldsymbol{v}}_2 = 6\overrightarrow{\boldsymbol{j}} + 6\overrightarrow{\boldsymbol{k}}$. The plane through $P_0 = (1,1,2)$ perpendicular to $6\overrightarrow{\boldsymbol{j}} + 6\overrightarrow{\boldsymbol{k}}$ is

$$0(x-2) + 6(y-1) + 6(z-2) = 0$$

$$\boxed{y + z = 3}.$$

Check.

Plug the equations for L_1 into the answer to see which t's cause a point on L_1 to also be on the proposed answer. The point on L_1 which corresponds to t is on the proposed answer provided $(2+t) + (1-t) = 3$. This happens all of the time. So L_1 is on the proposed answer. Similarly, The point on L_2 which corresponds to s is on the provided $(1+2s) + (2-2s) = 3$. This also happens all of the time. Both lines are on the proposed answer. The proposed answer is correct. ✓