12.3, number 19: **The picture (on the next page) makes it look like $\overrightarrow{v}_1 + \overrightarrow{v}_2$ and $\overrightarrow{v}_1 - \overrightarrow{v}_2$ are perpendicular. Does this happen all of the time? If not, what is special about the $\overrightarrow{v}_1$ and $\overrightarrow{v}_2$ in this picture that made it happen?**

Answer: The vectors $\overrightarrow{v}_1 + \overrightarrow{v}_2$ and $\overrightarrow{v}_1 - \overrightarrow{v}_2$ "appear" to be perpendicular . Let us check to see if they are or when they are.

Recall that two vectors are perpendicular precisely when their dot product is zero.

We take the dot product of $\overrightarrow{v}_1 + \overrightarrow{v}_2$ and $\overrightarrow{v}_1 - \overrightarrow{v}_2$ and see if it always is zero or what has to be special about $\overrightarrow{v}_1$ and $\overrightarrow{v}_2$ in for the dot product to be zero.

The dot product of $\overrightarrow{v}_1 + \overrightarrow{v}_2$ and $\overrightarrow{v}_1 - \overrightarrow{v}_2$ is

$$(\overrightarrow{v}_1 + \overrightarrow{v}_2) \cdot (\overrightarrow{v}_1 - \overrightarrow{v}_2) = \overrightarrow{v}_1 \cdot \overrightarrow{v}_1 - \overrightarrow{v}_1 \cdot \overrightarrow{v}_2 + \overrightarrow{v}_2 \cdot \overrightarrow{v}_1 - \overrightarrow{v}_2 \cdot \overrightarrow{v}_2.$$

Of course, $\overrightarrow{v}_1 \cdot \overrightarrow{v}_1$ is the same as the length of $\overrightarrow{v}_1$ squared; $\overrightarrow{v}_1 \cdot \overrightarrow{v}_2$ and $\overrightarrow{v}_2 \cdot \overrightarrow{v}_1$ are equal, so $-\overrightarrow{v}_1 \cdot \overrightarrow{v}_2 + \overrightarrow{v}_2 \cdot \overrightarrow{v}_1 = 0$; and $\overrightarrow{v}_2 \cdot \overrightarrow{v}_2$ is the same as the length of $\overrightarrow{v}_2$ squared. Thus,

$$(\overrightarrow{v}_1 + \overrightarrow{v}_2) \cdot (\overrightarrow{v}_1 - \overrightarrow{v}_2) = |\overrightarrow{v}_1|^2 - |\overrightarrow{v}_2|^2.$$

In other words, $\overrightarrow{v}_1 + \overrightarrow{v}_2$ and $\overrightarrow{v}_1 - \overrightarrow{v}_2$ are perpendicular precisely if $|\overrightarrow{v}_1|^2 = |\overrightarrow{v}_2|^2$ and this happens precisely when $\overrightarrow{v}_1$ and $\overrightarrow{v}_2$ have the same length.

We conclude that

If $\overrightarrow{v}_1$ and $\overrightarrow{v}_2$ are vectors, then the vectors $\overrightarrow{v}_1 + \overrightarrow{v}_2$ and $\overrightarrow{v}_1 - \overrightarrow{v}_2$ are perpendicular precisely when $\overrightarrow{v}_1$ and $\overrightarrow{v}_2$ have the same length.

The Picture for 12.3 #19