Math 174 Fall 2003 Final Exam.

PRINT Your Name:______________________________

There are 25 problems on 7 pages. Each problem is worth 4 points. The exam is worth 100 points. $\boxed{\textit{CIRCLE}}$ your answers. **No Calculators.**

WHEN YOU DO SOMETHING CLEVER, EXPLAIN YOUR WORK.

If I know your e-mail address, I will e-mail your course grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**. Otherwise, get your course grade from VIP.

I will post the solutions on my website later today.

You may leave the binomial coefficient $\binom{n}{r}$ in any of your answers.

1. Consider the relation " $\le$ " on the set of real numbers. Is this relation reflexive, symmetric, transitive? Explain.

2. If a and b are integers, then $a \equiv b \mod 5$ if and only if $5|(a-b)$. Describe the equivalance classes that correspond to this equivalence relation.

3. Suppose $n \equiv 1 \mod 5$. Give a formula for $\lfloor \frac{n}{5} \rfloor$ which does not involve $\lfloor\ \rfloor$.

4. Solve the recurrence relation $a_0 = 1000$, $a_n = (1.05)a_{n-1} + 100$ for $1 \le n$.

5. Consider the Tower of Hanoi problem. There are three towers in a row: tower A, tower B, and tower C. There are n disks of different sizes stacked on tower A. One must move all n disks to tower C. One may NEVER place a bigger disk on top of a smaller disk. In the present problem, one may move a disk only to an ADJACENT tower. Let a_n be the minimum number of moves needed to transfer a stack of n disks from tower A to tower C. Find a_1, a_2, a_3. Find a recurrence relation for $a_1, a_2, a_3, \dots$.

6. Does there exist a one-to-one and onto function from $\mathbb{N}$ to $\mathbb{N} \times \mathbb{N}$, where $\mathbb{N}$ is the set of positive integers? Explain.

7. Let r, m, and n be integers with $0 \le r \le m, n$. Simplify $\binom{n}{0}\binom{m}{r} + \binom{n}{1}\binom{m}{r-1} + \binom{n}{2}\binom{m}{r-2} + \cdots + \binom{n}{r-1}\binom{m}{1} + \binom{n}{r}\binom{m}{0}$. (Your answer should not have any $\dots$ or any summation signs.)

8. Let n be a positive integer. Simplify $\sum_{k=0}^{n} 2^k \binom{n}{k}$. (Your answer should not have any $\dots$ or any summation signs.)

9. How many four tuples (i, j, k, ℓ) are there with $3 \le i \le j \le k \le \ell \le 10$.

10. How many bit strings of length 8 contain exactly three 1's? (A bit string is a string of 0's and 1's.)

11. How many integers between 1 and 1000 are relatively prime to 28?

12. A group of eight people attend a movie together. John and Mary are part of the group and they refuse to sit next to one another. How many ways may the eight people be arranged in a row?

13. If the largest of 87 consecutive integers is 326, what is the smallest?

14. Let $A = \{t, u, v, w\}$ and let S_1 be the set of all subsets of A that do not contain w and S_2 the set of all subsets of A that do contain w.
    (a) Find S_1.
    (b) Find S_2.

15. Let $A = \{1, 2, 3\}$ and $B = \{4, 5\}$. List the elements of $A \times B$.

16. If A, B, and C are sets, is $A \cup (B \cap C) = (A \cup B) \cap C$? Prove or give a counterexample.

17. Prove $\sum\limits_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.

18. Consider the infinite sequence $a_1 = \frac{1}{4}$, $a_2 = \frac{2}{9}$, $a_3 = \frac{3}{16}$, $a_4 = \frac{4}{25}$, $a_5 = \frac{5}{36}$, $a_6 = \frac{6}{49}$, $\ldots$. What is a formula for a_n?

19. Prove that there are infinitely many prime integers.

20. True or False. The sum of two irrational numbers is irrational. Give a proof or a counterexample.

21. Prove that the square of any integer has the form $3k$ or $3k+1$ for some integer k.

22. True or False. If a, b, and c are integers with $a|bc$, then $a|b$ or $a|c$. Give a proof or a counterexample.

23. Write the repeating decimal $6.2\overline{34}$ as the ratio of two integers.

24. Write "Being divisible by 6 is a sufficient condition for being divisible by 3." in if-then form.

25. Prove that $p \rightarrow q$ is logically equivalent to the contrapositive of $p \rightarrow q$.