PRINT Your Name:______________________________

There are 11 problems on 4 pages. The exam is worth 50 points.

$\boxed{CIRCLE}$ your answers. **No Calculators.**

WHEN YOU DO SOMETHING CLEVER, EXPLAIN YOUR WORK.

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

If you would like, I will leave your exam outside my office door later today, you may pick it up any time between then and the next class. **Let me know if you are interested.**

I will post the solutions on my website at about 12:30 today.

You may leave the binbomial coefficient $\binom{n}{r}$ in any of your answers.

1. **(5 points) Flip a coin ten times, what is the proabability that the coin lands "Heads" exactly four times.**

Each outcome is a word of length in the letters H and T. For example HHHHHHHHHT represents the first 9 flips were H and the last flip is T. There are 2^{10} such words. A successful outcome has exactly 4 H's. There are $\binom{10}{4}$ ways to select 4 slots out of 10 slots. The final answer is $\boxed{\frac{\binom{10}{4}}{2^{10}}}$.

2. **(5 points) A committee consists of 10 people, 6 women and 4 men.**
   (a) How many subcommittees which consist of 6 people can be made?
   (b) How many subcommittees which consist of 4 women and 2 men can be made?

The answer to (a) is $\boxed{\binom{10}{6}}$. The answer to (b) is $\boxed{\binom{6}{4}\binom{4}{2}}$.

3. **(4 points) Six friends have pictures made. Each picture consists of 4 people arranged in a straight line? How many arrangements are possible?**

$\boxed{6 \cdot 5 \cdot 4 \cdot 3}$

4. **(4 points) How many integers between 1 and 1000 are relatively prime to 10?**

There are 1000 integers between 1 and 1000; 500 of these integers are even, we must throw them away; 200 of these integers are multiples of 5; we must not count them. We threw out every multiple of 10 twice; but we did want to count these so we bring back the 100 multiples of 10. Our answer is

$$\boxed{1000 - 500 - 200 + 100}.$$

5. **(5 points) Let $A = \{p, q, r, s\}$ and let S_1 be the set of all subsets of A that do not contain p and S_2 the set of all subsets of A that do contain p.**
   (a) Find S_1.
   (b) Find S_2.

$$\boxed{\begin{array}{c} S_1 = \{\emptyset, \{q\}, \{r\}, \{s\}, \{q,r\}, \{q,s\}, \{r,s\}, \{q,r,s\}\} \\ S_2 = \{\{p\}, \{p,q\}, \{p,r\}, \{p,s\}, \{p,q,r\}, \{p,q,s\}, \{p,r,s\}, \{p,q,r,s\}\} \end{array}}$$

6. **(4 points) True or False. If true, prove it. If false, then give a counterexample. For all sets A, B, and C, if $A \not\subseteq B$ and $B \not\subseteq C$, then $A \not\subseteq C$.**

$\boxed{\text{FALSE}}$. Let $A = C = \{1\}$ and $B = \{2\}$. We see that $A \not\subseteq B$ and $B \not\subseteq C$. We also see that $A \subseteq C$.

7. **(5 points) True or False. If true, prove it. If false, then give a counterexample. For all integers $n \geq 2$,**

$$\left(1 - \frac{1}{2^2}\right) \cdot \left(1 - \frac{1}{3^2}\right) \cdot \dots \cdot \left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}.$$

$\boxed{\text{TRUE}}$. We prove the formula by induction. When $n = 2$, the left side is $\left(1 - \frac{1}{2^2}\right) = \frac{3}{4}$ and the right side is $\frac{2+1}{2 \cdot 2} = \frac{3}{4}$. We assume as our induction hypothesis that

$$\text{(IH)} \qquad \left(1 - \frac{1}{2^2}\right) \cdot \left(1 - \frac{1}{3^2}\right) \cdot \dots \cdot \left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}$$

for some particular FIXED integer n, with $n \geq 2$. We will prove that

$$\text{(WWS)} \qquad \left(1 - \frac{1}{2^2}\right) \cdot \left(1 - \frac{1}{3^2}\right) \cdot \dots \cdot \left(1 - \frac{1}{n^2}\right) \cdot \left(1 - \frac{1}{(n+1)^2}\right) = \frac{n+2}{2n+2}.$$

We apply (IH) to see that the left side of (WWS) is

$$\frac{n+1}{2n} \cdot \left(1 - \frac{1}{(n+1)^2}\right) = \frac{n+1}{2n} \cdot \frac{(n+1)^2 - 1}{(n+1)^2} = \frac{1}{2n} \cdot \frac{n^2 + 2n}{n+1} = \frac{1}{2n} \cdot \frac{n(n+2)}{(n+1)}$$
$$= \frac{n+2}{2(n+1)},$$

which is the right side of (WWS). We have established that (WWS) is correct. The proof is complete.

8. **(5 points) True or False. If true, prove it. If false, then give a counterexample. For all integers $n \geq 1$,**

$$1 + 2^4 + 3^4 + \dots + n^4 = n^4 + n - 1.$$

$\boxed{\text{FALSE}}$. Take $n = 3$. The left side is $1 + 2^4 + 3^4 = 1 + 16 + 81 = 98$. The right side is $3^4 + 3 - 1 = 81 + 3 - 1 = 83$.

9. **(4 points) True or False. If true, prove it. If false, then give a counterexample. If $p_1, p_2, p_3, \ldots, p_r$ are prime integers, then $N = p_1 p_2 p_3 \cdots p_r + 1$ is a prime integer.**

$\boxed{\text{FALSE}}$. Take $r = 1$ and $p_1 = 3$. In this case $N = 3 + 1 = 4$, which is not prime.

10. **(4 points) Prove that there is no greatest even integer.**

Proof by contradiction. SUPPOSE that a greatest even integer did exist. Call this number N. Notice that $N + 2$ is even and $N + 2 > N$. But N is the greatest even integer. This is a contradiction. Everything we did was correct, except possibly our supposition. Our supposition must be false. We conclude that a greatest even integer does not exist.

11. **(5 points) How many permutations of a, b, c, d, e, f are there in which the first letter is a, b, c, or d and the last letter is c, d, e, or f.**

There are two cases. Either the first letter is a or b; or the first letter is c or d. If the first letter is a or b, then there are two choices for the first letter, 4 choices for the last letter and $4!$ choices for the intermediate letters. If the first letter is c or d, then there are 2 choices for the first letter, 3 choices for the last letter and $4!$ choices for the intermediate letters. The answer is

$$\boxed{2 \cdot 4! \cdot 4 + 2 \cdot 4! \cdot 3}$$