Math 174, Fall 2003, Solution to Quiz 8

Problem: If $f\colon X \to Y$ and $g\colon Y \to Z$ are functions and $g \circ f\colon X \to Z$ is one-to-one, must f and g both be one-to-one? Prove or give a counterexample.

Answer: NO! The function g does not have to be one-to-one. Consider $X = Z = \{1\}$, $Y = \{1, 2\}$, $f\colon X \to Y$ given by $f(1) = 1$, and $g\colon Y \to Z$ given by $g(1) = g(2) = 1$. Observe that $g \circ f\colon \{1\} \to \{1\}$, is $(g \circ f)(1) = 1$, which is one-to-one, but g is not one-to-one since $1 \neq 2$ and $g(1) = g(2)$.