MATH 142, EXAM 3, SPRING, 2004

PRINT Your Name:____________________________________

There are 10 problems on 5 pages. Each problem is worth 10 points. SHOW your work. $\boxed{CIRCLE}$ your answer. **NO CALCULATORS! CHECK** your answer whenever possible.

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

If you would like, I will leave your exam outside my office after I have graded it. (I will send you an e-mail when I am finished with it.) You may pick it up any time between then and the next class. **Let me know if you are interested.**

I will post the solutions on my website at about 4:00 PM today.

1. **Find** $\int \sin^3 x \cos^2 x \, dx$ **. Check your answer.**

Save one $\sin x$; convert the remaining $\sin^2 x$ to $1 - \cos^2 x$. The original problem is

$$\int (1 - \cos^2 x) \cos^2 x \sin x \, dx.$$

Let $u = \cos x$. It follows that $du = -\sin x \, dx$, and the integral is

$$-\int (1 - u^2) u^2 \, du = -\int (u^2 - u^4) \, du = -\left(\frac{u^3}{3} - \frac{u^5}{5}\right) + C$$

$$= \boxed{-\left(\frac{\cos^3 x}{3} - \frac{\cos^5 x}{5}\right) + C}.$$

Check. The derivative of the proposed answer is

$$-\big(\cos^2 x(-\sin x) - \cos^4 x(-\sin x)\big) = \sin x \cos^2 x(1 - \cos^2 x).\checkmark$$

2. **Find** $\int x \ln x \, dx$ **. Check your answer.**

Use integration by parts. Let $u = \ln x$ and $dv = x \, dx$. It follows that $du = \frac{dx}{x}$ and $v = \frac{x^2}{2}$. The original problem is

$$\int u \, dv = uv - \int v \, du = \frac{x^2 \ln x}{2} - \int \frac{x}{2} \, dx = \boxed{\frac{x^2 \ln x}{2} - \frac{x^2}{4} + C.}$$

Check. The derivative of the proposed answer is

$$\frac{x}{2} + x \ln x - \frac{x}{2}.\checkmark$$

3. **Find** $\int \frac{\ln x}{x} \, dx$ **. Check your answer.**

Let $u = \ln x$. It follows that $du = \frac{dx}{x}$. The original integral is equal to

$$\int u du = \frac{u^2}{2} + C = \boxed{\frac{(\ln x)^2}{2} + C.}$$

Check. The derivative of the proposed answer is

$$\ln x \cdot \frac{1}{x}.\checkmark$$

4. **Find** $\int \frac{4x^2-2x+1}{x(x^2+1)}\, dx$**. Check your answer.**

Write

$$\frac{4x^2-2x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}.$$

Multiply both sides by $x(x^2+1)$ to obtain

$$4x^2-2x+1 = A(x^2+1) + (Bx+C)x;$$

which is the same as

$$4x^2-2x+1 = (A+B)x^2 + Cx + A.$$

Equate the corresponding coefficients to see

$$4 = A+B, \quad -2 = C, \quad 1 = A.$$

So, $B = 3$. Check what we have so far:

$$\frac{1}{x} + \frac{3x-2}{x^2+1} = \frac{x^2+1+3x^2-2x}{x(x^2+1)} = \frac{4x^2-2x+1}{x(x^2+1)}.\checkmark$$

So, the original integral is equal to

$$\int \frac{1}{x} + \frac{3x-2}{x^2+1}\, dx = \boxed{\ln|x| + \frac{3}{2}\ln(x^2+1) - 2\arctan x + C.}$$

5. **Find** $\int \frac{x+1}{(x-1)^2}\, dx$**. Check your answer.**

Write

$$\frac{x+1}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2}.$$

Multiply both sides by $(x-1)^2$ to get

$$x+1 = A(x-1) + B;$$

that is,

$$x+1 = Ax + (B-A).$$

Equate the corresponding coefficients to conclude that $A=1$ and $B-A=1$; so, $B=2$. We check what we have so far:

$$\frac{1}{x-1}+\frac{2}{(x-1)^2}=\frac{x-1+2}{(x-1)^2}=\frac{x+1}{(x-1)^2}.\checkmark$$

So the original integral is

$$\int \frac{1}{x-1}+\frac{2}{(x-1)^2}\,dx=\boxed{\ln|x-1|-\frac{2}{x-1}+C.}$$

6. **Find** $\int\sqrt{1-x^2}\,dx$. **Check your answer.**

This integral contains an ugly a^2-u^2; so, I let $u=a\sin\theta$. That is, I let $x=\sin\theta$. It follows that $dx=\cos\theta\,d\theta$. Observe that

$$\sqrt{1-x^2}=\sqrt{1-\sin^2\theta}=\sqrt{\cos^2\theta}=\cos\theta.$$

The original integral is equal to

$$\int\cos^2\theta\,d\theta=\frac{1}{2}\int(1+\cos 2\theta)\;d\theta$$

$$=\frac{1}{2}\left(\theta+\frac{\sin 2\theta}{2}\right)+C=\frac{1}{2}\left(\arcsin x+\frac{2\sin\theta\cos\theta}{2}\right)+C$$

$$=\boxed{\frac{1}{2}\left(\arcsin x+x\sqrt{1-x^2}\right)+C.}$$

Check. The derivative of the proposed answer is

$$\frac{1}{2}\left(\frac{1}{\sqrt{1-x^2}}+\frac{-x^2}{\sqrt{1-x^2}}+\sqrt{1-x^2}\right)=\frac{1}{2}\left(\frac{1-x^2}{\sqrt{1-x^2}}+\sqrt{1-x^2}\right)$$

$$=\frac{1}{2}\left(\sqrt{1-x^2}+\sqrt{1-x^2}\right).\checkmark$$

7. **Find** $\lim\limits_{x\to 0}\dfrac{\cos x-1+\frac{x^2}{2}}{x^4}$ **.**

The top and the bottom both go to zero, so L'hopital's rule shows that the limit is

$$\lim_{x\to 0}\frac{-\sin x+x}{4x^3}.$$

The top and the bottom both go to zero, so L'hopital's rule shows that the limit is

$$\lim_{x\to 0}\frac{-\cos x+1}{12x^2}.$$

The top and the bottom both go to zero, so L'hopital's rule shows that the limit is

$$\lim_{x\to 0}\frac{\sin x}{24x}.$$

The top and the bottom both go to zero, so L'hopital's rule shows that the limit is

$$\lim_{x\to 0}\frac{\cos x}{24}=\boxed{\frac{1}{24}}.$$

8. **Find** $\lim\limits_{x\to 0}\frac{\cos x}{x-2}$ **.**

The top goes to 1 and the bottom goes to -2. The limit is $\boxed{\dfrac{1}{-2}}$.

9. **Find the limit of the sequence whose n^{th} term is $a_n = \left(\frac{n-1}{n+1}\right)^n$.**

We must find

$$\lim_{n\to\infty}\left(\frac{n-1}{n+1}\right)^n.$$

This limit has the indeterminate form that the base goes to 1 and the exponent goes to ∞. Let $y = \left(\frac{n-1}{n+1}\right)^n$. We must find $\lim\limits_{n\to\infty} y$. We do find

$$\lim_{n\to\infty}\ln y = \lim_{n\to\infty} n\ln\left(\frac{n-1}{n+1}\right) = \lim_{n\to\infty}\frac{\ln\left(\frac{n-1}{n+1}\right)}{\frac{1}{n}}.$$

The top and the bottom both go to 0. Apply L'hopital's rule to get that

$$\lim_{n\to\infty}\ln y = \lim_{n\to\infty}\frac{\frac{1}{n-1}-\frac{1}{n+1}}{\frac{-1}{n^2}} = \lim_{n\to\infty}\frac{-2n^2}{n^2-1} = -2.$$

So, the answer is

$$\lim_{n\to\infty} y = \lim_{n\to\infty} e^{\ln y} = \boxed{e^{-2}}.$$

One could also do the problem by observing that

$$\lim_{n\to\infty}\left(\frac{n-1}{n+1}\right)^n = \lim_{n\to\infty}\left(\frac{(n+1)-2}{n+1}\right)^n = \lim_{n\to\infty}\left(1+\frac{-2}{n+1}\right)^n.$$

Let $m = n+1$. This limit is equal to

$$\lim_{m\to\infty}\left(1+\frac{-2}{m}\right)^{m-1} = \lim_{m\to\infty}\frac{\left(1+\frac{-2}{m}\right)^m}{1+\frac{-2}{m}}.$$

We know that

$$\lim_{m\to\infty}\left(1+\frac{r}{m}\right)^m = e^r;$$

so, the limit of the top is e^{-2} and the limit of the bottom is 1. Once again, we get the answer e^{-2}.

10. **Find $\int_{-1}^{3}\frac{1}{x^2}\,dx$.**

The function $f(x) = \frac{1}{x^2}$ goes to infinity as x approaches zero. For all x other than zero, $f(x)$ is positive. Thus, this integral represents an area. Either this integral is infinite; or else, the integral is finite and positive.

$$\int_{-1}^{3}\frac{1}{x^2}\,dx = \lim_{b\to 0^-}\int_{-1}^{b}\frac{1}{x^2}\,dx + \lim_{a\to 0^+}\int_{a}^{3}\frac{1}{x^2}\,dx$$

$$= \lim_{b\to 0^-}\left.\frac{-1}{x}\right|_{-1}^{b} + \lim_{a\to 0^+}\left.\frac{-1}{x}\right|_{a}^{3} = \lim_{b\to 0^-}\left(\frac{-1}{b}-\frac{-1}{-1}\right) + \lim_{a\to 0^+}\left(\frac{-1}{3}-\frac{-1}{a}\right).$$

Notice that

$$\lim_{b\to 0^-}\left(\frac{-1}{b}\right) = +\infty \quad\text{and}\quad \lim_{a\to 0^+}\left(-\frac{-1}{a}\right) = +\infty.$$

Thus, $\boxed{\text{the integral diverges to } +\infty.}$ Notice that $-\frac{4}{3}$ has NOTHING TO DO WITH THE FINAL ANSWER. An "answer" $-\frac{4}{3}$ will recieve a score of 0.