Math 142, Exam 4, Fall 2006, Solutions

Write your answers as legibly as you can on the blank sheets of paper provided.

Please leave room in the upper left corner for the staple.

Use only **one side** of each sheet. Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.; although, by using enough paper, you can do the problems in any order that suits you.

The exam is worth a total of 100 points. There are 10 problems. Each problem is worth 10 points.

SHOW your work. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible. **No Calculators or Cell phones.**

I will post the solutions on my website sometime this afternoon.

If I know your e-mail address, I will e-mail your grade to you as soon as the exam is graded. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

1. **Write** $12.457575757\dots$ **as a quotient of two integers.**

Let $N = 12.457575757\dots$. It follows that

$$99N = 100N - N = 1245.7575757\cdots - 12.457575757 = 1245.7 - 12.4 = 1233.3$$

Thus, $N = \frac{1233.3}{99} = \boxed{\dfrac{12333}{990}}$.

2. **Approximate** $\sum\limits_{k=1}^{\infty} \frac{1}{k^4}$ **with an error at most** $\frac{1}{2000}$. **Explain very thoroughly.**

Let $S = \sum\limits_{k=1}^{\infty} \frac{1}{k^4}$ and $s_n = \sum\limits_{k=1}^{n} \frac{1}{k^4}$. We use s_n, for large enough n, to approximate S. It is clear that

$$|S - s_n| = \sum_{k=n+1}^{\infty} \frac{1}{k^4}.$$

Draw a picture to see that

$$|S - s_n| = \sum_{k=n+1}^{\infty} \frac{1}{k^4} \le \int_n^{\infty} \frac{1}{x^4} dx = \lim_{b\to\infty} \frac{1}{-3x^3}\bigg|_n^b = \lim_{b\to\infty} \frac{1}{-3b^3} + \frac{1}{3n^3} = \frac{1}{3n^3}.$$

We make n large enough for $\frac{1}{3n^3} \le \frac{1}{2000}$. That is, we want $\frac{2000}{3} \le n^3$. Notice that $\frac{2000}{3} < 729 = 9^3$. We conclude that

$$\boxed{\sum_{k=1}^{9} \frac{1}{k^4} \text{ approximates } \sum_{k=n+1}^{\infty} \frac{1}{k^4} \text{ with an error less than } \frac{1}{2000}.}$$

3. **Give a closed formula for $3^2+3^3+3^4+\cdots+3^{99}$. There should be no dots or summation signs in your answer. Your answer should be exactly equal to the given number.**

Let $S = 3^2 + 3^3 + 3^4 + \cdots + 3^{99}$. It follows that

$$2S = 3S - S = 3^3 + 3^4 + \cdots + 3^{99} + 3^{100} - (3^2 + 3^3 + 3^4 + \cdots + 3^{99}) = 3^{100} - 3^2.$$

We conclude that

$$\boxed{S = \frac{3^{100} - 3^2}{2}.}$$

4. **Does $\sum\limits_{k=1}^{\infty} \frac{4+\cos^2 k}{k^3}$ converge? Explain very thoroughly.**

Compare $\sum\limits_{k=1}^{\infty} \frac{4+\cos^2 k}{k^3}$ to the convergent p-series $5\sum\limits_{k=1}^{\infty} \frac{1}{k^3}$ (we have $1 < 3 = p$.) It is clear that $\frac{4+\cos^2 k}{k^3} \le \frac{5}{k^3}$. We conclude that

$$\boxed{\sum_{k=1}^{\infty} \frac{4+\cos^2 k}{k^3} \text{ converges.}}$$

5. **Does** $\sum\limits_{k=1}^{\infty} \frac{k^2}{2^k}$ **converge? Explain very thoroughly.**

Apply the ratio test. We see that

$$\rho = \lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{(k+1)^2}{2^{k+1}} \frac{2^k}{k^2} = \lim_{k\to\infty} \frac{(1+\frac{1}{k})^2}{2} = \frac{1}{2} < 1.$$

We conclude that

$$\boxed{\sum_{k=1}^{\infty} \frac{k^2}{2^k} \text{ converges.}}$$

6. **Does** $\sum\limits_{k=1}^{\infty} \frac{2}{k+5}$ **converge? Explain very thoroughly.**

Apply the limit comparison test with the divergent Harmonic series $\sum\limits_{k=1}^{\infty} \frac{1}{k}$. We see that

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\frac{2}{k+5}}{\frac{1}{k}} = \lim_{k\to\infty} \frac{2k}{k+5} = \lim_{k\to\infty} \frac{2}{1+\frac{5}{k}} = 2.$$

We know that 2 is a number; it is not 0 or ∞ . We conclude that

$$\boxed{\sum_{k=1}^{\infty} \frac{2}{k+5} \text{ also diverges.}}$$

7. **Does** $\sum\limits_{k=1}^{\infty} \left(\frac{k-5}{k}\right)^k$ **converge? Explain very thoroughly.**

We use the Individual Term test For Divergence. We see that

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \left(\frac{k-5}{k}\right)^k = \lim_{k\to\infty} \left(1 + \frac{-5}{k}\right)^k = e^{-5}.$$

We know that e^{-5} is not zero. We conclude that

$$\boxed{\sum_{k=1}^{\infty} \left(\frac{k-5}{k}\right)^k \text{ diverges.}}$$

8. **Consider the sequence $\{a_n\}$ with $a_1 = \sqrt{30}$, and $a_n = \sqrt{30 + a_{n-1}}$ for $n \geq 2$. Prove that the sequence $\{a_n\}$ is increasing. Identify an upper bound for the sequence $\{a_n\}$. Prove that your candidate for an upper bound really is an upper bound. Find the limit of the sequence $\{a_n\}$. Explain very thoroughly.**

We show that the sequence is bounded by 6 by induction. It is clear that $a_1 < 6$. Assume that $a_n < 6$ for some fixed n. We now have

$$a_{n+1} = \sqrt{30 + a_n} < \sqrt{30 + 6} = 6.$$

We conclude that $a_n < 6$ for all n.

We show that the sequence is an increasing sequence. We know that $a_{n-1} < 6$. It follows that $a_{n-1} - 6 < 0$. Multiply both sides by the positive number $a_n + 5$. We know that $(a_{n-1} - 6)(a_n + 5) < 0$. Expand to see that $a_{n-1}^2 - a_{n-1} - 30 < 0$. Add $a_{n-1} + 30$ to both sides to see that $a_{n-1}^2 < 30 + a_{n-1}$. The square root function is an increasing function; so, $a_{n-1} < \sqrt{30 + a_{n-1}} = a_n$. We have shown that $a_{n-1} < a_n$ for all n. We conclude that the sequence $\{a_n\}$ is an increasing sequence.

The completeness axiom guarantees that the sequence converges to some limit; called, say L. Take the limit of both sides of $a_n = \sqrt{30 + a_{n-1}}$ to see that $L = \sqrt{30 + L}$. Square both sides to see that $L^2 = 30 + L$, or $L^2 - L - 30 = 0$. In other words, $(L - 6)(L + 5) = 0$. It follows that $L = 6$ or $L = -5$. Every number in the sequence is positive. Thus, L can not be negative. We conclude that $L = 6$.

9. **Find a closed formula for the n^{th} partial sum of the series**

$$\ln\left(1 - \frac{1}{4}\right) + \ln\left(1 - \frac{1}{9}\right) + \ln\left(1 - \frac{1}{16}\right) + \dots .$$

Explain very thoroughly. Be sure to do the problem that I picked – and not some other problem.

I count as follows:

$$s_1 = \ln\left(1 - \frac{1}{4}\right),$$

$$s_2 = \ln\left(1-\frac{1}{4}\right) + \ln\left(1-\frac{1}{9}\right),$$

$$\begin{aligned}
s_n &= \ln\left(1-\frac{1}{4}\right) + \ln\left(1-\frac{1}{9}\right) + \cdots + \ln\left(1-\frac{1}{(n+1)^2}\right) \\
&= \ln\left(\frac{3}{4}\right) + \ln\left(\frac{8}{9}\right) + \cdots + \ln\left(\frac{(n+1)^2-1}{(n+1)^2}\right) \\
&= \ln\left(\frac{1\cdot 3}{2\cdot 2}\right) + \ln\left(\frac{2\cdot 4}{3\cdot 3}\right) + \cdots + \ln\left(\frac{[(n+1)-1][(n+1)+1]}{(n+1)\cdot(n+1)}\right) \\
&= \left(\ln\frac{1}{2}+\ln\frac{3}{2}\right) + \left(\ln\frac{2}{3}+\ln\frac{4}{3}\right) + \left(\ln\frac{3}{4}+\ln\frac{5}{4}\right) + \cdots + \left(\ln\frac{n}{n+1}+\ln\frac{n+2}{n+1}\right) \\
&= \ln\frac{1}{2} + \left(\ln\frac{3}{2}+\ln\frac{2}{3}\right) + \left(\ln\frac{4}{3}+\ln\frac{3}{4}\right) + \left(\ln\frac{5}{4}+\cdots+\ln\frac{n}{n+1}\right) + \ln\frac{n+2}{n+1} \\
&\boxed{= \ln\frac{1}{2} + \ln\frac{n+2}{n+1}}.
\end{aligned}$$

10. **Find all values of x for which the series:**

$$\frac{1}{x^2} + \frac{2}{x^3} + \frac{4}{x^4} + \frac{8}{x^5} + \frac{16}{x^6} + \ldots$$

converges. What is the sum of the series? Explain very thoroughly.

The given series is the Geometric series with initial term $a = \frac{1}{x^2}$ and ratio $r = \frac{2}{x}$. If $|r| < 1$, then the series converges to $\frac{a}{1-r}$. If $\frac{2}{|x|} < 1$,then the series converges to $\frac{\frac{1}{x^2}}{1-\frac{2}{x}}$.

$$\boxed{\text{If } 2 < |x|, \text{ then the series converges to } \tfrac{1}{x^2-2x}.}$$