**Math 142, Exam 3, Fall 2006**

Write your answers as legibly as you can on the blank sheets of paper provided.

**Please leave room in the upper left corner for the staple.**

Use only **one side** of each sheet. Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.; although, by using enough paper, you can do the problems in any order that suits you.

The exam is worth a total of 100 points. There are 10 problems. Each problem is worth 10 points.

SHOW your work. $\boxed{\mathit{CIRCLE}}$ your answer. **CHECK** your answer whenever possible. **No Calculators or Cell phones.**

I will post the solutions on my website sometime this afternoon.

If I know your e-mail address, I will e-mail your grade to you as soon as the exam is graded. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

1. **A cone shaped water reservoir is 20 feet in diameter across the top and 15 feet deep. If the reservoir is filled to a depth of 10 feet, how much work is required to pump all the water to the top of the reservoir? The density of water is $62.4\,\mathrm{lb/ft}^3$. (There is no need for you to do arithmetic. Leave your answer in terms of sums and products of numbers.)**

Look at the picture. Notice that my axis is aimed downward, with $y = 0$ at the top. The RADIUS of the top is 10. The radius of the thin layer of water whose $y$-coordinate is $y$ is $10 - \frac{2}{3}y$. Draw similar triangles or merely find the equation of the line that includes the points $r = 10$, $y = 0$ and $r = 0$, $y = 15$. The work to lift the layer with $y$-cordinate $y$ is the weight of the layer times the distance this layer is to be lifted, and this is

$$\text{vol} \times \text{density} \times \text{distance}$$

$$= \pi r^2 t(62.4)y = \pi\left(10 - \frac{2}{3}y\right)^2 (62.4)y\,dy,$$

with $t = dy$. At the beginning the water water covers from $y = 5$ to $y = 15$. The work required to pump all of the water out of the tank is

$$\int_5^{15} \pi \left(10 - \frac{2}{3}y\right)^2 y(62.4))dy$$

$$= (62.4)\pi \int_5^{15} \left(100y - \frac{40}{3}y^2 + \frac{4}{9}y^3\right) dy$$

$$\boxed{= (62.4)\pi \left(50y^2 - \frac{40}{9}y^3 + \frac{1}{9}y^4\right)\Big|_5^{15} \text{ foot-pounds.}}$$

2. **Find** $\displaystyle\int_0^5 \frac{1}{(x-4)^2}\,dx$ **.**

Draw the picture. Notice that the graph is always above the $x$-axis and that the graph goes to $\infty$ as $x$ approaches $4$. Hence the answer is either some positive number or else the integral diverges to $+\infty$. The integral is equal to

$$\lim_{b\to 4^-} \int_0^b \frac{1}{(x-4)^2}dx + \lim_{a\to 4^+} \int_a^5 \frac{1}{(x-4)^2}dx$$

$$= \lim_{b\to 4^-} \frac{-1}{(x-4)}\Big|_0^b + \lim_{a\to a^+} \frac{-1}{x-4}\Big|_a^5$$

$$= \lim_{b\to 4^-} \left(\frac{-1}{b-4} - \frac{1}{4}\right) + \lim_{a\to 4^+} \left(-1 + \frac{1}{a-4}\right).$$

Both limits are $+\infty$. $\boxed{\text{The integral diverges to } +\infty.}$ (The number $-\frac{5}{4}$ has nothing to do with this problem. If your answer is $-\frac{5}{4}$, then your score on this problem will be $0$.)

3. **Find** $\int \sin^3 x\,dx$. **Check your answer.**

THe integral is $\int (1-\cos^2 x)\sin x\,dx$. Let $u = \cos x$. It follows that $du = -\sin x\,dx$ and the integral is

$$-\int (1-u^2)du = -\left(u - \frac{u^3}{3}\right) + C = \boxed{-\cos x + \frac{\cos^3 x}{3} + C}.$$

<u>Check</u> The derivative of the proposed answer is $\sin x - \cos^2 x \sin x$ $= \sin x(1-\cos^2 x)\checkmark$ .

4. **Find** $\int \sin^2 x\,dx$.

The integral is equal to

$$\frac{1}{2}\int (1-\cos 2x)\,dx = \boxed{\frac{1}{2}\left(x - \frac{\sin(2x)}{2}\right) + C}$$

5. **Find** $\int \sec^3 x\,dx$. **Check your answer.**

Let $u = \sec x$ and $dv = \sec^2 x\,dx$. It follows that $du = \sec x \tan x\,dx$ and $v = \tan x$. The integration by parts formula gives:

$$\int \sec^3 x dx = \sec x \tan x - \int \sec x \tan^2 x\,dx$$

$$= \sec x \tan x - \int \sec x(\sec^2 x - 1)\,dx$$

$$= \sec x \tan x - \int \sec^3 x\,dx + \ln|\sec x + \tan x|.$$

Add $\int \sec^3 x\,dx$ to both sides to see that

$$2\int \sec^3 x\,dx = \sec x \tan x + \ln|\sec x + \tan x|.$$

Divided by 2 to conclude

$$\boxed{\int \sec^3 x\,dx = \frac{1}{2}\left(\sec x \tan x + \ln|\sec x + \tan x|\right) + C}$$

<u>Check</u> The derivative of the proposed answer is

$$\frac{1}{2}(\sec x \sec^2 x + \sec x \tan^2 x + \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}) = \frac{1}{2}\left(\sec^3 x + \sec x(1 + \tan^2 x)\right)$$

$$= \sec^3 x \checkmark$$

6. **Find** $\int \frac{-2x^2 - 2x + 1}{x^2(x^2+1)}\,dx$. **Check your answer.**

Use the technique of partial fractions to see that

$$\frac{-2x^2 - 2x + 1}{x^2(x^2+1)} = \frac{-2}{x} + \frac{1}{x^2} + \frac{2x-3}{x^2+1}.$$

The integral is equal to

$$\boxed{-2\ln|x| - \frac{1}{x} + \ln(x^2+1) - 3\arctan x + C.}$$

7. **What is the limit of the sequence whose** $n^{\text{th}}$ **term is** $a_n = \left(\frac{n+5}{n}\right)^n$ **?**

We see that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{n+5}{n}\right)^n = \lim_{n\to\infty}\left(1 + \frac{5}{n}\right)^n = \boxed{e^5}.$$

I used $\lim\limits_{n\to\infty}(1 + \frac{r}{n})^n = e^r$ .

8. **Does the integral $\int_2^\infty \frac{x}{x^6+1}\,dx$ converge? Explain your answer very thoroughly.**

We see that $0 \le \frac{x}{x^6+1}$, for all $x$ under consideration. There are only two possibilities. Either $\int_2^\infty \frac{x}{x^6+1}\,dx$ is a finite number or $\int_2^\infty \frac{x}{x^6+1}\,dx$ goes to infinity. We see that $\frac{x}{x^6+1} \le \frac{x}{x^6} = \frac{1}{x^5}$. It follows that

$$\int_2^\infty \frac{x}{x^6+1}\,dx = \lim_{b\to\infty} \int_2^b \frac{x}{x^6+1}\,dx \le \lim_{b\to\infty} \int_2^b \frac{1}{x^5}\,dx = \lim_{b\to\infty} \left.\frac{1}{-4x^4}\right|_2^b$$

$$= \lim_{b\to\infty} \frac{1}{-4b^4} - \frac{1}{-4(2)^4} = \frac{1}{64}.$$

$$\boxed{\text{Thus, } \int_2^\infty \frac{x}{x^6+1}\,dx \text{ converges to some finite non-negative number which is at most } \frac{1}{64}.}$$

9. **Consider the sequence**

$$\begin{aligned} a_1 &= \sqrt{6} \\ a_2 &= \sqrt{6+\sqrt{6}} \\ a_3 &= \sqrt{6+\sqrt{6+\sqrt{6}}} \\ a_4 &= \sqrt{6+\sqrt{6+\sqrt{6+\sqrt{6}}}} \\ &\vdots \end{aligned}$$

**Assume that you know that the sequence converges. Find the limit of the sequence. Explain your work very thoroughly.**

Let $L$ be the name of $\lim\limits_{n\to\infty} a_n$. We see that $a_{n+1} = \sqrt{6+a_n}$. Take the limit of each side to see that $L = \sqrt{L+6}$. Square both sides to get $L^2 = L+6$, or $L^2 - L - 6 = 0$. Factor to get $(L-3)(L+2) = 0$. It follows that $L$ must equal $-2$ or $3$. However, all of the members of our sequence are at least $0$; so $L \ge 0$. Thus, $L \ne -2$ and $\boxed{L = 3}$.

10. **Trapezoidal rule says that** $\int_a^b f(x)dx = T_n + E_n$ **. The formula for** $T_n$ **is not needed in this problem. The value of** $E_n$ **is** $E_n = \frac{-(b-a)^3}{12n^2} f''(c)$ **for some** $c$ **between** $a$ **and** $b$ **. Suppose that Trapezoidal rule is used to approximate** $\int_1^3 \frac{1}{x}\,dx$ **, with** $n = 10$ **. Find a decent upper bound for** $|E_n|$ **. Explain your work very thoroughly.**

We have $f(x) = x^{-1}$ , $f'(x) = -x^{-2}$ , and $f''(x) = 2x^{-3}$ . So, if $n = 10$ , then

$$|E_n| = |\frac{-(b-a)^3}{12n^2} f''(c)| = |\frac{-(3-1)^3}{12(10)^2} \frac{2}{c^3}| = \frac{8}{1200} \frac{2}{c^3}.$$

If $1 \le c \le 3$ , then $\frac{1}{3^3} \le \frac{1}{c^3} \le \frac{1}{1}$ , thus,

$$|E_n| \le \frac{16}{1200} = \frac{1}{75}.$$