Math 142, Exam 1, Fall 2006

Write your answers as legibly as you can on the blank sheets of paper provided.

Please leave room in the upper left corner for the staple.

Use only **one side** of each sheet. Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.; although, by using enough paper, you can do the problems in any order that suits you.

The exam is worth a total of 100 points. There are 10 problems. Each problem is worth 10 points.

SHOW your work. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible. **No Calculators or Cell phones.**

I will post the solutions on my website sometime Wednesday afternoon.

I will grade the exam Wednesday afternoon. If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

1. **Define the definite integral. Give a complete definition. Be sure to explain all of your notation.**

Let $f(x)$ be a function defined on the closed interval $a \le x \le b$. For each partition P of the closed interval $[a, b]$ (so, P is $a = x_0 \le x_1 \le \cdots \le x_n = b$), let $\Delta_i = x_i - x_{i-1}$, and pick $x_i^* \in [x_{i-1}, x_i]$. The definite integral $\int_a^b f(x)dx$ is the limit over all partitions P as all Δ_i go to zero of $\sum_{i=1}^{n} f(x_i^*)\Delta_i$.

2. **State both parts of the Fundamental Theorem of Calculus. Be sure to explain all of your notation.**

Let f be a continuous function defined on the closed interval $[a, b]$.

(a) If $A(x)$ is the function $A(x) = \int_a^x f(t)dt$, for all $x \in [a, b]$, then $A'(x) = f(x)$ for all $x \in [a, b]$.

(b) If $F(x)$ is any antiderivative of $f(x)$, then $\int_a^b f(x)dx = F(b) - F(a)$.

3. **Find** $\displaystyle\int_0^1 \frac{y^2\,dy}{\sqrt{4-3y}}$.

Let $u = 4-3y$. We see that $du = -3dy$. When $y=0$, then $u=4$. When $y=1$, then $u=1$. The original integral is equal to

$$\frac{-1}{3}\int_4^1 \frac{\left(\frac{4-u}{3}\right)^2 du}{\sqrt{u}} = \frac{-1}{27}\int_4^1 \frac{16-8u+u^2}{\sqrt{u}}\,du = \frac{-1}{27}\int_4^1 (16u^{-1/2}-8u^{1/2}+u^{3/2})\,du.$$

$$\boxed{= \frac{-1}{27}\left(32u^{1/2} - \frac{16}{3}u^{3/2} + \frac{2}{5}u^{5/2}\right)\Bigg|_4^1}$$

4. **Find** $\displaystyle\int_0^1 \frac{x\,dx}{\sqrt{4-3x^4}}$.

The integral is equal to $\displaystyle\frac{1}{2}\int_0^1 \frac{x\,dx}{\sqrt{1-\frac{3x^4}{4}}}$. Let $u = \frac{\sqrt{3}x^2}{2}$. It follows that $du = \sqrt{3}x\,dx$. When $x=0$, then $u=0$. When $x=1$, then $u=\frac{\sqrt{3}}{2}$. The integral is equal to

$$\frac{1}{2\sqrt{3}}\int_0^{\frac{\sqrt{3}}{2}} \frac{du}{\sqrt{1-u^2}} = \frac{1}{2\sqrt{3}}\arcsin u\Big|_0^{\frac{\sqrt{3}}{2}} = \boxed{\frac{1}{2\sqrt{3}}\left(\frac{\pi}{3}\right)}.$$

5. **Find** $\lim\limits_{x\to\infty}\left(\frac{x-2}{x}\right)^{3x}$**.**

This limit is equal to $\left(\lim\limits_{x\to\infty}\left(1+\frac{-2}{x}\right)^x\right)^3$. I know that $\lim\limits_{x\to\infty}\left(1+\frac{r}{x}\right)^x = e^r$. It follows that this limit is $\left(e^{-2}\right)^3 = \boxed{e^{-6}}$.

6. **Find the area between** $x=y^2$ **and** $8=x+2y$**.**

I drew a picture else where. I partition the y-axis from $y=-4$ to $y=2$. I approximate the area using boxes of base dy and length $(8-2y)-y^2$. The area is

$$\int_{-4}^2 8-2y-y^2\,dy = \boxed{\left(8y-y^2-\frac{y^3}{3}\right)\Bigg|_{-4}^2}.$$

7. **Find the volume of the solid whose base is the region bounded between the curves $y = x$ and $y = x^2$, and whose cross sections perpendicular to the x-axis are squares.**

I drew a picture else where. I partition the x-axis from 0 to 1. I create one slice of my solid for each part of my partition. This slice has thickness dx, base $x - x^2$, and height $x - x^2$. The volume is

$$\int_0^1 (x - x^2)^2 dx = \int_0^1 x^2 - 2x^3 + x^4 dx = \boxed{\left(\frac{x^3}{3} - \frac{x^4}{2} + \frac{x^5}{5}\right)\Big|_0^1}$$

8. **Consider the region bounded by $y = x^2$, the x-axis, and $x = 1$. Rotate the region about the x-axis. Set up an integral which will give the volume of the resulting solid. You do not have to evaluate the integral. You do have to explain your work.**

I drew a picture elsewhere. Rotate the rectangle . Get a disk of volume $\pi r^2 t$, where $t = dx$ and $r = x^2$. The volume is

$$\boxed{\pi \int_0^1 x^4\, dx}.$$

9. **Consider the region bounded by $y = x^2$, the x-axis, and $x = 1$. Rotate the region about the y-axis. Set up an integral which will give the volume of the resulting solid. You do not have to evaluate the integral. You do have to explain your work.**

I drew a picture elsewhere. Rotate the rectangle . Get a cylindrical shell of volume $2\pi rht$, where $t = dx$, $r = x$, and $h = x^2$. The volume is

$$\boxed{2\pi \int_0^1 x^3\, dx}.$$

10. **Consider the region bounded by $y = x^2$, the x-axis, and $x = 1$. Rotate the region about the line $x = -5$. Set up an integral which will give the volume of the resulting solid. You do not have to evaluate the integral. You do have to explain your work.**

I drew a picture elsewhere. Rotate the rectangle . Get a cylindrical shell of volume $2\pi rht$, where $t = dx$, $r = x + 5$, and $h = x^2$. The volume is

$$\boxed{2\pi \int_0^1 (x+5)x^2\, dx}.$$