PRINT Your Name:____________________________________

Math 142 Exam 4 Fall 2004 Solutions

There are 10 problems on 5 pages. Each problem is worth 10 points. SHOW your work. $\boxed{CIRCLE}$ your answer. **NO CALCULATORS!**

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

If you would like, I will leave your exam outside my office after I have graded it. (I will send you an e-mail when I am finished with it.) You may pick it up any time between then and the next class. **Let me know if you are interested.**

I will post the solutions on my website at about 6:00 PM today.

1. **What is the limit of the sequence:**

$$\sin 1,\ 2\sin\frac{1}{2},\ 3\sin\frac{1}{3},\ 4\sin\frac{1}{4},\ 5\sin\frac{1}{5},\ \ldots?$$

Explain your answer.

We compute

$$\lim_{n\to\infty} n\sin\frac{1}{n} = \lim_{n\to\infty}\frac{\sin\frac{1}{n}}{\frac{1}{n}}.$$

The top and the bottom both go to zero, so L'hopital's rule yields that the above limit is

$$= \lim_{n\to\infty}\frac{-n^{-2}\cos\frac{1}{n}}{-n^{-2}} = \lim_{n\to\infty}\cos\frac{1}{n} = \cos 0 = \boxed{1}.$$

2. **Does the series $\sum_{n=2}^{\infty}\frac{3^n}{4^{n+1}}$ converge? Explain your answer.**

This series $\boxed{\text{converges}}$ because it is the geometric series with ratio $r=\frac{3}{4}$ and initila term $a=\frac{9}{64}$. We notice that $-1<r<1$.

3. **Does the series $\sum_{n=2}^{\infty}\frac{n}{n^2-1}$ converge? Explain your answer.**

We make a straight comparison with the divergent Harmonic series $\sum_{n=2}^{\infty}\frac{1}{n}$. Notice that $n^2-1\le n^2$. Divide both sides by $n(n^2-1)$ to see that $\frac{1}{n}\le\frac{1}{n^2-1}$. We conclude that the series $\sum_{n=2}^{\infty}\frac{n}{n^2-1}$ also $\boxed{\text{diverges}}$.

4. **Does the series** $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ **converge? Explain your answer.**

We apply the integral test. Notice that $f(x)$ is a positive decreasing function for $2 \le x$. We compute

$$\int_2^{\infty} \frac{1}{x \ln x} dx = \lim_{b\to\infty} \ln|\ln x| \Big|_2^b = \lim_{b\to\infty} \ln|\ln b| - \ln\ln 2 = +\infty.$$

The integral diverges. Therefore, the series also $\boxed{\text{diverges}}$.

5. **Does the series**

$$1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \frac{1}{5\sqrt{5}} + \ldots$$

converge? Explain your answer.

This series is the p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$, with $p = 3/2$. We see that $1 < p$. We conclude that the series $\boxed{\text{converges}}$.

6. **For which values of** x **does the power series** $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^n}{n2^n}$ **converge? Explain your answer.**

Let

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{(x-3)^{n+1}}{(n+1)2^{n+1}}}{\frac{(x-3)^n}{n2^n}}\right| = \lim_{n\to\infty} \frac{|x-3|^{n+1}}{(n+1)2^{n+1}} \frac{n2^n}{|x-3|^n}$$

$$= \lim_{n\to\infty} \frac{|x-3|}{2} \frac{n}{(n+1)} = \frac{|x-3|}{2}.$$

We see that $\rho < 1$ when $\frac{|x-3|}{2} < 1$, or $|x-3| < 2$, which is $-2 < x-3 < 2$, or $1 < x < 5$. We also see that $1 < \rho$ for $x < 1$ or $5 < x$. At $x = 5$, $f(5) = \sum_{n=1}^{\infty} \frac{2^n}{n2^n} = \sum_{n=1}^{\infty} \frac{1}{n}$, and this series is the divergent Harmonic series. At $x = 1$, $f(1) = \sum_{n=1}^{\infty} \frac{(-2)^n}{n2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, and this series is minus the convergent Alternating Harmonic series. We conclude that

$$\boxed{f(x) \text{ converges for } 1 \le x < 5 \text{ and } f(x) \text{ diverges for all other } x.}$$

7. **Approximate** $\int_0^1 \sin(x^2)\,dx$ **with an error of at most** $\frac{1}{10^4}$ **. Explain your answer.**

We know that

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\ .$$

Replace every x by x^2 to get

$$\sin x^2 = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \dots\ .$$

So,

$$\int_0^1 \sin(x^2)\,dx = \int_0^1 x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \dots\ dx$$

$$= \frac{x^3}{3} - \frac{x^7}{3!7} + \frac{x^{11}}{5!11} - \frac{x^{15}}{7!15} + \dots \Big|_0^1 = \frac{1}{3} - \frac{1}{3!7} + \frac{1}{5!11} - \frac{1}{7!15} + \dots$$

Notice that this series is an alternating series which satisfies the hypotheses of the Alternating Series Test because the terms (in absolute value) are decreasing and going to zero. Notice also that $7!(15) = 120(6)(7)(15) > 10^4$, so $\frac{1}{7!15} < \frac{1}{10^4}$ and the Alternating series test assures us that $\int_0^1 \sin(x^2)\,dx$ may be approximated by

$$\boxed{\frac{1}{3} - \frac{1}{3!7} + \frac{1}{5!11}}$$

with an error of no more than $\frac{1}{10^4}$.

8. **What familiar series is equal to**

$$x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \frac{x^{10}}{5!} + \dots ?$$

Explain your answer.

We know that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots\ .$$

It follows that

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \frac{x^{10}}{5!} + \dots\ ,$$

and

$$\boxed{e^{x^2} - 1} = x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \frac{x^{10}}{5!} + \dots\ .$$

9. **Find the Taylor Polynomial $P_3(x)$ for the function $f(x) = \sqrt{x}$ about $a = 1$.**

We know that

$$P_3(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3.$$

We compute that

$$f(x) = \sqrt{x}, \quad f'(x) = \frac{1}{2\sqrt{x}}, \quad f''(x) = \frac{-1}{4x^{3/2}}, \quad f'''(x) = \frac{3}{8x^{5/2}};$$

and therefore,

$$f(1) = 1, \quad f'(1) = \frac{1}{2}, \quad f''(1) = \frac{-1}{4}, \quad f'''(1) = \frac{3}{8}.$$

We conclude that

$$\boxed{P_3(x) = 1 + \frac{(x-1)}{2} - \frac{1}{8}(x-1)^2 + \frac{3}{3!8}(x-1)^3.}$$

10. **Give an upper bound for the difference between $\sum\limits_{n=1}^{10} \frac{1}{n^4}$ and $\sum\limits_{n=1}^{\infty} \frac{1}{n^4}$. I expect your upper bound to be relatively small and correct. Be sure to explain what you are doing and why you are allowed to do it.**

The function $f(x) = \frac{1}{x^4}$ is positive and decreasing; so, we may use the enclosed picture to see that the area under the curve is greater than or equal to the area inside the boxes:

$$\left| \sum_{n=1}^{\infty} \frac{1}{n^4} - \sum_{n=1}^{10} \frac{1}{n^4} \right| = \sum_{n=11}^{\infty} \frac{1}{n^4} = \text{area inside the boxes} \le \text{area under the curve}$$

$$= \int_{10}^{\infty} \frac{1}{x^4}\,dx = \lim_{b\to\infty} \left.\frac{-1}{3x^3}\right|_{10}^{b} = \lim_{b\to\infty} \left(\frac{-1}{3b^3} + \frac{1}{3(10)^3} \right) = \frac{1}{3(10)^3}.$$

We conclude that $\sum\limits_{n=1}^{\infty} \frac{1}{n^4}$ may be approximated by $\sum\limits_{n=1}^{10} \frac{1}{n^4}$ with an error of at most $\boxed{\frac{1}{3000}}$.