Math 142, Exam 4, Solutions, Spring 2006

There are 10 problems. Each problem is worth 10 points. Write in complete sentences. JUSTIFY EVERY ASNSWER **VERY THOROUGHLY**.

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

I will post the solutions on my website a few hours after the exam is finished.

1. **Does the series** $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2+\frac{1}{k}}$ **converge? Justify your answer.**

Observe that the limit of the k^{th} term is $\lim\limits_{k\to\infty} \frac{1}{2+\frac{1}{k}} = \frac{1}{2}$, which is not zero. The series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2+\frac{1}{k}}$ DIVERGES by the Divergence Test.

2. **Does the series** $\displaystyle\sum_{k=1}^{\infty} \frac{k}{2^k}$ **converge? Justify your answer.**

Use the Ratio Test. Compute that

$$\rho = \lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = \lim_{k\to\infty} \frac{k+1}{2^{k+1}}\frac{2^k}{k} = \lim_{k\to\infty} \frac{1+\frac{1}{k}}{2} = \frac{1}{2}.$$

We see that $\rho < 1$; so the ratio test guarantees that $\sum\limits_{k=1}^{\infty} \frac{k}{2^k}$ CONVERGES.

3. **Does the series** $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$ **converge? Justify your answer.**

We apply the alternating series test. We are looking at an alternating series. The terms in absolute value are decreasing to zero. Indeed, $\frac{1}{\sqrt{k}} > \frac{1}{\sqrt{k+1}}$ and $\lim\limits_{k\to\infty} \frac{1}{\sqrt{k}} = 0$. The Alternating Series Test applies. We conclude that $\sum\limits_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$ CONVERGES.

4. **Does the series** $\displaystyle\sum_{k=1}^{\infty} \frac{\arctan k}{k^2}$ **converge? Justify your answer.**

Use the comparison test. We know that $\arctan k \le \frac{\pi}{2}$, for all k. Hence, $\frac{\arctan k}{k^2} \le \frac{\frac{\pi}{2}}{k^2}$. The series $\sum\limits_{k=1}^{\infty} \frac{1}{k^2}$ is the p-series with $p = 2$, which is bigger than 1. We know that $\sum\limits_{k=1}^{\infty} \frac{1}{k^2}$ converges, and therefore, the series $\frac{\pi}{2} \sum\limits_{k=1}^{\infty} \frac{1}{k^2}$ also converges. We conclude that $\sum\limits_{k=1}^{\infty} \frac{\arctan k}{k^2}$ also CONVERGES by the comparison Test.

5. **Does the series** $\displaystyle\sum_{k=2}^{\infty} \frac{1}{\sqrt{k(k-1)}}$ **converge? Justify your answer.**

We do a limit comparison with the divergent Harmonic series $\sum\limits_{k=1}^{\infty} \frac{1}{k}$. We compute the limit as k goes to infinity of the quotient of the k^{th} term from one series divided by the k^{th} term from the other series:

$$\lim_{k\to\infty} \frac{\frac{1}{\sqrt{k(k-1)}}}{\frac{1}{k}}$$

$$= \lim_{k\to\infty} \frac{k}{\sqrt{k(k-1)}} = \lim_{k\to\infty} \frac{1}{\sqrt{1-\frac{1}{k}}} = 1.$$

The above limit is a number, not 0 and not ∞. Thus, both series $\sum\limits_{k=1}^{\infty} \frac{1}{k}$ and $\sum\limits_{k=1}^{\infty} \frac{1}{\sqrt{k(k-1)}}$ both converge or both diverge. We have already noticed that $\sum\limits_{k=1}^{\infty} \frac{1}{k}$ diverges. We conclude that $\sum\limits_{k=1}^{\infty} \frac{1}{\sqrt{k(k-1)}}$ also DIVERGES.

6. **Consider the sequence $\{a_n\}$ with $a_1 = \sqrt{20}$, and $a_n = \sqrt{20 + a_{n-1}}$ for $n \geq 2$. Prove that the sequence $\{a_n\}$ converges. Find the limit of the sequence $\{a_n\}$.**

Notice that $a_n \leq 5$ for all n. It is clear that $a_1 < 5$. If $a_{n-1} \leq 5$, then $a_{n-1} + 20 \leq 25$; hence, $a_n = \sqrt{a_{n-1}+20} \leq \sqrt{25} = 5$. Our assertion is established by Mathematical Induction.

We now claim that the sequence $\{a_n\}$ is an increasing sequence. We know that $a_{n-1} \leq 5$. Multiply both sides by the positive number $a_{n-1} + 4$ to see that $a_{n-1}^2 + 4a_{n-1} \leq 5a_{n-1} + 20$. In other words, $a_{n-1}^2 \leq a_{n-1} + 20$. The square root function is an increasing function; so, $a_{n-1} \leq \sqrt{a_{n-1}+20} = a_n$. Our claim is established.

The sequence $\{a_n\}$ is an increasing sequence which never gets beyond 5. The Completeness axiom guarantees that the sequence converges. Let L be the name of $\lim\limits_{n\to\infty} a_n$. Take the limit of $a_n = \sqrt{20 + a_{n-1}}$ to see that $L = \sqrt{L+20}$. We can now solve for L. We have $L^2 = L + 20$ or $L^2 - L - 20 = 0$. We factor to get $(L-5)(L+4) = 0$. So, $L = -4$ or $L = 5$. All of our a_n are positive so L can not possibly negative. We conclude that $\boxed{L = 5}$.

7. **Approximate the sum $\sum\limits_{k=1}^{\infty} \frac{1}{k^3}$ with an error of at most $\frac{1}{100}$. Justify your answer.**

Let S be the sum of the entire series $S = \sum\limits_{k=1}^{\infty} \frac{1}{k^3}$ and s_n be the partial sum $s_n = \sum\limits_{k=1}^{n} \frac{1}{k^3}$. It is clear that

$$|S - s_n| = \sum_{k=n+1}^{\infty} \frac{1}{k^3}.$$

We will find a decent overestimate of $\sum\limits_{k=n+1}^{\infty} \frac{1}{k^3}$. We will make our overestimate of the error be less than $\frac{1}{100}$. Then we will know that the error is at most $\frac{1}{100}$. I drew some boxes on another page. The area inside the boxes is less than the area under the curve. The area inside the boxes is $\sum\limits_{k=n+1}^{\infty} \frac{1}{k^3}$. The area under the curve is

$$\int_n^{\infty} \frac{1}{x^3} dx = \lim_{b\to\infty} \frac{1}{-2x^2}\bigg|_n^b = \lim_{b\to\infty} \frac{1}{2n^2} - \frac{1}{2b^2} = \frac{1}{2n^2}.$$

We make $\frac{1}{2n^2} \le \frac{1}{100}$. We make $50 \le n^2$. We make $8 \le n$. We conclude that $\sum\limits_{k=1}^{8} \frac{1}{k^3}$ approximates $\sum\limits_{k=1}^{\infty} \frac{1}{k^3}$ with an error at most $\frac{1}{100}$.

8. **Approximate the sum $\sum\limits_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3}$ with an error of at most $\frac{1}{100}$. Justify your answer.**

We apply the Alternating series Test. The series is alternating. The terms (in absolute value) are decreasing to zero. We conclude that the series converges. Let S be the sum of the entire series $S = \sum\limits_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3}$ and let s_n be the n^{th} partial sum $s_n = \sum\limits_{k=1}^{n} \frac{(-1)^{k+1})}{k^3}$. The alternating series test tells us that

$$|S - s_n| \le a_{n+1} = \frac{1}{(n+1)^3}.$$

We take n large enough to have

$$\frac{1}{(n+1)^3} \leq \frac{1}{100}$$

We want $100 \leq (n+1)^3$. We may take $n = 4$. We conclude that $\sum_{k=1}^{4} \frac{(-1)^{k+1}}{k^3}$ approximates $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3}$ with an error at most $\frac{1}{100}$.

9. **A ball is dropped from the height of 40 feet. Each time it hits the floor it rebounds to $\frac{5}{7}$ its previous height. Find the total distance it travels. Explain what you are doing.**

The most interesting part of the problem is that the first event is only down. Every other event is a round trip: up and then down. The distance the ball travels is

$$40 + \tfrac{5}{7}(40) + \tfrac{5}{7}(40) + (\tfrac{5}{7})^2 40 + (\tfrac{5}{7})^2 40 + \dots$$

$$= 40 + \frac{5}{7}(80)\left[1 + \frac{5}{7} + \left(\frac{5}{7}\right)^2 + \left(\frac{5}{7}\right)^3 + \dots\right] = \boxed{40 + \frac{5}{7}(80)\frac{1}{1-\frac{5}{7}}.}$$

We used the fact that the geometric series with initial term a and ratio r converges to $\frac{a}{1-r}$ whenever $-1 < r < 1$.

10. **Give a closed formula for $s_n = \sum_{k=2}^{n} \ln\left(1 - \frac{1}{k^2}\right)$. (Your formula should be exactly equal to the sum I have given. Your formula should not contain any dots or any summation signs.) Explain what you are doing.**

We see that $s_n = \sum_{k=2}^{n} \ln \frac{k^2-1}{k^2} = \sum_{k=2}^{n} \ln\left(\frac{(k+1)(k-1)}{k^2}\right) = \sum_{k=2}^{n} \left[\ln\left(\frac{k-1}{k}\right) + \ln\left(\frac{k+1}{k}\right)\right]$

$$= \left(\ln\frac{1}{2} + \ln\frac{3}{2}\right) + \left(\ln\frac{2}{3} + \ln\frac{4}{3}\right) + \left(\ln\frac{3}{4} + \ln\frac{5}{4}\right) + \dots + \left(\ln\frac{n-1}{n} + \ln\frac{n+1}{n}\right)$$

$$= \ln\frac{1}{2} + \left(\ln\frac{3}{2} + \ln\frac{2}{3}\right) + \left(\ln\frac{4}{3} + \ln\frac{3}{4}\right) + \dots + \left(\ln\frac{n}{n-1} + \ln\frac{n-1}{n}\right) + \ln\frac{n+1}{n}$$

$$= \boxed{\ln\frac{1}{2} + \ln\frac{n+1}{n}}.$$