Math 142, Exam 2, Spring 2014

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: return the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. $\boxed{CIRCLE}$ your answer.

No Calculators or Cell phones.

1. (9 points) **Rotate the region between $y = 5 - x^2$ and $y = x^2 - 3$ about the line $x = -6$. Find the volume of the resulting solid. You must draw a meaningful picture. Write in complete sentences. Your work must be coherent, complete, and correct.**

The curves intersect when $5 - x^2 = x^2 - 3$. Thus, the curves intersect when $8 = 2x^2$. Thus, the points of intersection are $(-2, 1)$ and $(2, 1)$. We drew a picture elsewhere. Chop the x-axis from $x = -2$ to $x = 2$. For each little piece of the x-axis, draw a rectangle from the lower parabola to the higher porabola. Consider the rectangle with x-coordinate x. Rotate the rectangle. Get a shell of volume $2\pi rht$, where $t = dx$, $r = x + 6$, and $h = 8 - 2x^2$. (Look at the picture to see the value of these parameters.) The volume of one shell is $2\pi rht = 2\pi(x+6)(8-2x^2)dx$. The volume of the solid is

$$2\pi \int_{-2}^{2} (x+6)(8-2x^2)dx = 2\pi \int_{-2}^{2} (-2x^3 - 12x^2 + 8x + 48)dx$$

$$= 2\pi \left(-\frac{x^4}{2} - 4x^3 + 4x^2 + 48x \right)\Bigg|_{-2}^{2}$$

$$= 2\pi(-8 - 32 + 16 + 96 - (-8 + 32 + 16 - 96)) = \boxed{4(64)\pi}.$$

2. (9 points) **Let $S = \sum\limits_{k=2}^{27} 3^k$. Find a closed formula for S. Write in complete sentences. Your work must be coherent, complete, and correct. (Answer the question that I asked. Keep in mind that a closed formula does not have any summation signs or any dots.)**

We see that $3S - S = 3^{28} - 3^2$; so $\boxed{S = (3^{28} - 3^2)/2}$.

3. (8 points) **Approximate** $\sum\limits_{k=1}^{\infty} \frac{1}{k^8}$ **with an error at most** $\frac{1}{7\cdot 10^7}$ **. Write in complete sentences. Your work must be coherent, complete, and correct.**

The series $\sum\limits_{k=1}^{\infty} \frac{1}{k^8}$ is the p-series with $p = 8 > 1$. This series converges. We approximate $\sum\limits_{k=1}^{\infty} \frac{1}{k^8}$ with $\sum\limits_{k=1}^{N} \frac{1}{k^8}$, for some carefully chosen N. We used the picture on the other page to see that

$$\left| \sum_{k=1}^{\infty} \frac{1}{k^8} - \sum_{k=1}^{N} \frac{1}{k^8} \right| \le \frac{1}{7N^7}$$

We want our approximation to have error at most $\frac{1}{7\cdot 10^7}$; so we pick N large enough that $\frac{1}{7N^7} \le \frac{1}{7\cdot 10^7}$. Obviously, every N with $10 \le N$ is large enough. We conclude that

$$\boxed{\sum_{k=1}^{10} \frac{1}{k^8} \text{ approximates } \sum_{k=1}^{\infty} \frac{1}{k^8} \text{ with an error at most } \frac{1}{7\cdot 10^7}}$$

4. (8 points) **Does** $\sum\limits_{k=1}^{\infty} \frac{1}{k+k\cos^2 k}$ **converge? Justify your answer. Write in complete sentences. Your work must be coherent, complete, and correct.**

We see that $k \le k + k\cos^2 k \le 2k$. It follows that $\frac{1}{2k} \le \frac{1}{k+k\cos^2 k} \le \frac{1}{k}$. The series $\sum\limits_{k=1}^{\infty} \frac{1}{2k}$ is one half of the harmonic series. The harmonic series diverges; so one half of the harmonic series also diverges. Thus, $\boxed{\sum\limits_{k=1}^{\infty} \frac{1}{k+k\cos^2 k} \text{ diverges}}$ by the comparison test.

5. (8 points) **Does** $\sum\limits_{k=1}^{\infty} k(\frac{2}{3})^k$ **converge? Justify your answer. Write in complete sentences. Your work must be coherent, complete, and correct.**

Use the ratio test. Let

$$\rho = \lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{(k+1)(\frac{2}{3})^{k+1}}{k(\frac{2}{3})^k} = \lim_{k\to\infty} (1+\tfrac{1}{k})(\tfrac{2}{3}) = \tfrac{2}{3} < 1.$$

We conclude that $\boxed{\sum_{k=1}^{\infty} k(\frac{2}{3})^k \text{ converges}}$ by the ratio test.

6. (8 points) **Does** $\sum_{k=1}^{\infty} \frac{2+(-1)^k}{k\sqrt{k}}$ **converge? Justify your answer. Write in complete sentences. Your work must be coherent, complete, and correct.**

We see that $\frac{1}{k^{3/2}} < \frac{2+(-1)^k}{k\sqrt{k}} < \frac{3}{k^{3/2}}$. The series $\sum_{k=1}^{\infty} \frac{3}{k^{3/2}}$ is 3 times the p-series $\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$ with $p = 3/2 > 1$. The series $\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$ converges; thus, $\sum_{k=1}^{\infty} \frac{3}{k^{3/2}}$ converges, and

$$\boxed{\sum_{k=1}^{\infty} \frac{2+(-1)^k}{k\sqrt{k}} \text{ converges}}$$

by the comparison test.