**Math 142, Exam 2, Solutions, Spring 2006**

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.; although, by using enough paper, you can do the problems in any order that suits you.

There are 10 problems. Each problem is worth 10 points. SHOW your work. Make your work be coherent and clear. Write in complete sentences whenever this is possible. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible. **No Calculators.**

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

I will post the solutions on my website a few hours after the exam is finished.

1. **Find** $\int \cos^3 x dx$ **.**

The integral is equal to

$$\int (1-\sin^2 x)\cos x dx.$$

Let $u = \sin x$ . It follows that $du = \cos x dx$ and the integral is equal to

$$\int (1-u^2)du = u - \frac{u^3}{3} + C = \boxed{\sin x - \frac{\sin^3 x}{3} + C}.$$

Check. The derivative of the proposed answer is

$$\cos x - \sin^2 x \cos x = \cos x(1-\sin^2 x) = \cos^3 x. \checkmark$$

2. **Find** $\int \cos^4 x dx$ **.**

The integral is equal to

$$\int \cos^4 x dx = \int \left(\frac{1+\cos 2x}{2}\right)^2 dx = \frac{1}{4}\int (1+2\cos 2x + \cos^2 2x)dx$$

$$= \frac{1}{4}\int (1 + 2\cos 2x + \frac{1+\cos 4x}{2})dx = \frac{1}{8}\int (2 + 4\cos 2x + 1 + \cos 4x)dx$$

$$= \frac{1}{8}\int (3 + 4\cos 2x + \cos 4x)dx = \boxed{\frac{1}{8}\left(3x + 2\sin 2x + \frac{\sin 4x}{4}\right) + C}.$$

Check. The derivative of the proposed answer is

$$\frac{1}{8}\left(3+4\cos 2x+\cos 4x\right)=\frac{1}{8}\left(2+4\cos 2x+(1+\cos 4x)\right)$$

$$=\frac{1}{8}\left(2+4\cos 2x+2\cos^2 x\right)=\frac{2}{8}\left(1+2\cos 2x+\cos^2 x\right)$$

$$=\frac{1}{4}\left(1+\cos^2 x\right)^2=\left(\frac{1+\cos^2 x}{2}\right)^2=\cos^4 x. \checkmark$$

3. **Find** $\int \tan^3 x dx$ **.**

The integral is equal to

$$\int(\sec^2 x-1)\tan x dx=\int \sec^2 x\tan x dx-\int\frac{\sin x}{\cos x}dx$$

$$=\boxed{\frac{\tan^2 x}{2}+\ln|\cos x|+C}.$$

Check. The derivative of the proposed answer is

$$\tan x\sec^2 x-\frac{\sin x}{\cos x}=\tan x(\sec^2 x-1)=\tan^3 x. \checkmark$$

4. **Find** $\int \sin 4x\cos 3x dx$ **.**

Add

$$\begin{aligned}\sin(A+B)&=\sin A\cos B+\cos A\sin B\\ \sin(A-B)&=\sin A\cos B-\cos A\sin B\end{aligned}$$

to see that

$$\frac{1}{2}(\sin(A+B)+\sin(A-B))=\sin A\cos B.$$

Let $A=4x$ and $B=3x$ to see that the original problem is equal to

$$\frac{1}{2}\int(\sin 7x+\sin x)dx=\boxed{\frac{1}{2}(\frac{-\cos 7x}{7}-\cos x)+C.}$$

5. **Find** $\int xe^x dx$ **.**

Use integration by parts with $u = x$ and $dv = e^x dx$ . It follows that $du = dx$ and $v = e^x$ . The integration by parts formula is

$$\int udv = uv - \int vdu.$$

Our integral is equal to

$$xe^x - \int e^x dx = \boxed{xe^x - e^x + C.}$$

Check. The derivative of the proposed answer is

$$xe^x + e^x - e^x = xe^x. \checkmark$$

6. **Find** $\int xe^{x^2} dx$ **.**

The integral is equal to

$$\boxed{\tfrac{1}{2}e^{x^2} + C}$$

Check. The derivative of the proposed answer is

$$\tfrac{1}{2}2xe^{x^2} = xe^{x^2}. \checkmark$$

7. **State BOTH parts of the Fundamental Theorem of Calculus.**

Let $f$ be a continuous function defined on the closed interval $[a, b]$ .
(a) If $A(x)$ is the function $A(x) = \int_a^x f(t)dt$ , for all $x \in [a, b]$ , then $A'(x) = f(x)$ for all $x \in [a, b]$ .
(b) If $F(x)$ is any antiderivative of $f(x)$ , then $\int_a^b f(x)dx = F(b) - F(a)$.

8. **Find** $\lim\limits_{x\to\infty} \left(\frac{x-2}{x}\right)^x$ **.**

This limit is equal to $\lim\limits_{x\to\infty} \left(1 + \frac{-2}{x}\right)^x$ . I know that $\lim\limits_{x\to\infty} (1 + \frac{r}{x})^x = e^r$ . Apply this fact with $r = -2$ to see that the limit is $\boxed{e^{-2}}$. (There are other ways to do the problem which require less knowledge and more calculation.)

9. **Consider the region bounded by $y = \ln x$, the $x$-axis, $x = 1$, and $x = 2$. Revolve this region about the $y$-axis. Find the volume of the resulting solid?**

I drew a picture elsewhere. I partition the $x$-axis. Over each subinterval I draw a rectangle. Spin the rectangle. Get a shell of volume $2\pi rht$, with $t = dx$, $r = x$, and $h = \ln x$. It follows that the volume is

$$2\pi \int_1^2 x \ln x dx.$$

Use integration by parts with $u = \ln x$, $dv = xdx$. It follows that $du = \frac{1}{x}dx$, $v = \frac{x^2}{2}$, and the volume is equal to

$$2\pi \left( \ln x \frac{x^2}{2} - \frac{1}{2}\int xdx \right)\Bigg|_1^2$$

$$2\pi \left( \ln x \frac{x^2}{2} - \frac{x^2}{4} \right)\Bigg|_1^2$$

$$= \boxed{2\pi(2\ln 2 - 1 - 0 + \frac{1}{4}).}$$

Check the integral. The derivative of

$$\ln x \frac{x^2}{2} - \frac{x^2}{4}$$

is

$$x \ln x + \frac{x}{2} - \frac{x}{2} = x \ln x. \checkmark$$

10. **Suppose that a conical tank is filled with oil, which has a density of 50 lb/ft$^3$. The radius at the top of the tank is 6 feet and the tank is 20 feet high. Set up an integral which gives the work which is done in pumping the oil over the edge of the tank. You are not required to calculate the integral. (Be sure to give the units for your answer.)**

I drew a picture elsewhere. Notice how I arranged my axis. I put $x = 0$ at the top of the tank, and $x = 20$ at the bottom of the tank. I will have to lift the oil that starts at $x$-coordinate $x$, $x$ feet. I partition the $x$-axis, and figure out how much work is required to lift the thin layer of oil at $x$-coordinate $x$. I add up the work required for each layer, and then I take the limit, making the thickness of each layer go to zero. In other words, I integrate.

Anyhow, the layer of oil at $x$-coordinate $x$ has volume $\pi r^2 t$, where $t$ is $dx$ and $r$ depends on $x$. The relationship between $r$ and $x$ is linear. When $x = 0$, then $r = 6$. When $x = 20$, then $r = 0$. So, $r = \frac{-3}{10}x + 6$. (One can also do this game using similar triangles.) The weight of the thin layer of oil is its volmue times its density:

$$\pi\left(\frac{-3}{10}x + 6\right)^2 (50)dx,$$

and this is measured in pounds. We must lift this thin layer of oil $x$ feet. The total work is

$$\boxed{50\pi \int_0^{20} \left(\frac{-3}{10}x + 6\right)^2 x dx}$$

and the answer is given in foot-pounds.