Math 142, Exam 1, Solutions Fall 2012

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: return the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. SHOW your work. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible.

No Calculators or Cell phones.

The solutions will be posted later today.

1. (7 points) **Define the definite integral. Give a complete definition. Be sure to explain all of your notation.**

Let $f(x)$ be a function defined on the closed interval $a \le x \le b$. For each partition P of the closed interval $[a,b]$ (so, P is $a = x_0 \le x_1 \le \cdots \le x_n = b$), let $\Delta_i = x_i - x_{i-1}$, and pick $x_i^* \in [x_{i-1}, x_i]$. The definite integral $\int_a^b f(x)dx$ is the limit over all partitions P as all Δ_i go to zero of $\sum\limits_{i=1}^{n} f(x_i^*)\Delta_i$.

2. (7 points) **State both parts of the Fundamental Theorem of Calculus. Be sure to explain all of your notation.**

Let f be a continuous function defined on the closed interval $[a,b]$.

(a) If $A(x)$ is the function $A(x) = \int_a^x f(t)dt$, for all $x \in [a,b]$, then $A'(x) = f(x)$ for all $x \in [a,b]$.

(b) If $F(x)$ is any antiderivative of $f(x)$, then $\int_a^b f(x)dx = F(b) - F(a)$.

3. (7 points) **Find** $\displaystyle\int \frac{e^{\frac{1}{x}}}{x^2}\,dx$. **Check your answer.**

Answer: Let $u = \frac{1}{x}$. Compute $du = \frac{-1}{x^2}dx$. The original integral is equal to

$$= -\int e^u du = -e^u + C = \boxed{-e^{\frac{1}{x}} + C}.$$

Check: The derivative of the proposed answer is

$$-(-1)e^{\frac{1}{x}}\frac{1}{x^2}. \checkmark$$

4. (7 points) **Find** $\int (\ln x)^2\, dx$ **. Check your answer.**

Answer: Try integration by parts: $\int u dv = uv - \int v du$. Let $u = (\ln x)^2$ and $dv = dx$. It follows that $du = 2\ln x(\frac{1}{x})dx$ and $v = x$. The original integral is

$$x(\ln x)^2 - 2\int \ln x dx.$$

Use integration by parts again. Let $u = \ln x$ and $dv = dx$. It follows that $du = (\frac{1}{x})dx$ and $v = x$. The original integral is

$$x(\ln x)^2 - 2(x\ln x - \int dx) = \boxed{x(\ln x)^2 - 2(x\ln x - x) + C}.$$

Check . The derivative of the proposed answer is

$$x2\ln x\left(\frac{1}{x}\right) + (\ln x)^2 - 2\left(x\left(\frac{1}{x}\right) + \ln x - 1\right) = 2\ln x + (\ln x)^2 - 2\left(1 + \ln x - 1\right)$$

$$= (\ln x)^2. \checkmark$$

5. (7 points) **Find** $\int \sin 8x \cos 5x\, dx$ **. Check your answer.**

Answer: Add the two identities

$$\begin{aligned}\sin(A+B) &= \sin A\cos B + \cos A\sin B\\ \sin(A-B) &= \sin A\cos B - \cos A\sin B\end{aligned}$$

to get

$$\sin(A+B) + \sin(A-B) = 2\sin A\cos B.$$

Divide by 2 to get

$$\tfrac{1}{2}\left[\sin(A+B) + \sin(A-B)\right] = \sin A\cos B.$$

Take $A = 8x$ and $B = 5x$. We have

$$\int \sin 8x\cos 5x\, dx = \frac{1}{2}\int \left[\sin(13x) + \sin(3x)\right] dx = \boxed{\frac{-\cos(13x)}{26} - \frac{\cos(3x)}{6} + C}.$$

6. (7 points) **Find** $\int \dfrac{dx}{\sqrt{x^2+16}}$ **. Check your answer.**

Answer: Let $x = 4\tan\theta$. Compute $dx = 4\sec^2\theta d\theta$ and

$$x^2 + 16 = 16\tan^2\theta + 16 = 16\sec^2\theta.$$

The original integral is

$$\int \frac{4\sec^2\theta d\theta}{4\sec\theta} = \int \sec\theta d\theta = \ln|\sec\theta + \tan\theta| + C = \ln\left|\frac{\sqrt{x^2+16}}{4} + \frac{x}{4}\right| + C$$

$$= \ln\left|\sqrt{x^2+16} + x\right| - \ln(4) + C = \boxed{\ln\left|\sqrt{x^2+16} + x\right| + K},$$

where K is the constant $K = \ln(4) + C$.

Check: The derivative of the proposed answer is

$$\frac{\frac{2x}{2\sqrt{x^2+16}} + 1}{\sqrt{x^2+16} + x} = \frac{\left(\frac{x}{\sqrt{x^2+16}} + 1\right)\sqrt{x^2+16}}{(\sqrt{x^2+16} + x)\sqrt{x^2+16}} = \frac{x + \sqrt{x^2+16}}{(\sqrt{x^2+16} + x)\sqrt{x^2+16}}$$

$$= \frac{1}{\sqrt{x^2+16}}. \checkmark$$

7. (8 points) **Find** $\int \sqrt{5 + 4x - x^2}\, dx$. **Check your answer.**

Answer: We complete the square: $5+4x-x^2 = 5+4-(x^2-4x+4) = 9-(x-2)^2$. Let $x - 2 = 3\sin\theta$. It follows that $9 - (x-2)^2 = 9 - 9\sin^2\theta = 9\cos^2\theta$ and $dx = 3\cos\theta d\theta$. The original problem is equal to

$$\int 3\cos\theta(3\cos\theta)\, d\theta = 9\int \cos^2\theta\, d\theta = \frac{9}{2}\int (1 + \cos 2\theta) d\theta = \frac{9}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + C$$

$$= \frac{9}{2}\left(\theta + \frac{2\sin\theta\cos\theta}{2}\right) + C = \frac{9}{2}\left(\arcsin\left(\frac{x-2}{3}\right) + \frac{x-2}{3}\frac{\sqrt{5+4x-x^2}}{3}\right) + C$$

$$= \boxed{\frac{9}{2}\arcsin\left(\frac{x-2}{3}\right) + \frac{1}{2}(x-2)\sqrt{5+4x-x^2} + C.}$$

<ins>Check</ins>. The derivative of the proposed answer is

$$\frac{9}{2}\frac{\frac{1}{3}}{\sqrt{1 - \frac{(x-2)^2}{9}}} + \frac{1}{2}\left(\sqrt{5+4x-x^2} + (x-2)\frac{4-2x}{2\sqrt{5+4x-x^2}}\right)$$

$$= \frac{1}{2}\left(\frac{3}{\frac{1}{3}\sqrt{9-(x-2)^2}} + \sqrt{5+4x-x^2} + (x-2)\frac{2-x}{\sqrt{5+4x-x^2}}\right)$$

$$= \frac{1}{2}\left(\frac{9}{\sqrt{5+4x-x^2}} + \sqrt{5+4x-x^2} + \frac{-(x-2)^2}{\sqrt{5+4x-x^2}}\right)$$

$$= \frac{1}{2}\left(\frac{9-(x-2)^2}{\sqrt{5+4x-x^2}} + \sqrt{5+4x-x^2}\right) = \frac{1}{2}\left(\sqrt{5+4x-x^2} + \sqrt{5+4x-x^2}\right)$$

$$= \sqrt{5+4x-x^2} \ \checkmark$$