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Quiz for August 31, 2005

Let $f(x) = 3x^2 + 5x - 2$ for $0 \le x$.
a. Find $f^{-1}(x)$.
b. Find the domain of $f^{-1}(x)$.
c. Verify that $f(f^{-1}(x)) = x$ for all x in the domain of $f^{-1}(x)$.
d. Verify that $f^{-1}(f((x)) = x$ for all x in the domain of $f(x)$.

ANSWER: Let $y = f^{-1}(x)$. We know that $f(y) = x$ and that y is in the domain of f. In other words, $3y^2 + 5y - 2 = x$ and $0 \le y$. Re-write the equation to get $3y^2 + 5y - 2 - x = 0$. This is a quadratic equation of the form $ay^2 + by + c = 0$, with $a = 3$, $b = 5$, and $c = -2 - x$. Apply the quadratic formula. The solution of $ay^2 + by + c = 0$ is $y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. So for us,

$$y = \frac{-5 \pm \sqrt{25 - 12(-2-x)}}{6} = \frac{-5 \pm \sqrt{49 + 12x}}{6}$$

Our y MUST be at least zero; so we need, to take $+$ rather than $-$. Also, we must have

$$\begin{gathered}
0 \le \frac{-5 + \sqrt{49 + 12x}}{6} \\
0 \le -5 + \sqrt{49 + 12x} \\
5 \le \sqrt{49 + 12x} \\
25 \le 49 + 12x \\
-24 \le 12x \\
-2 \le x.
\end{gathered}$$

Our answer to (a) is $\boxed{f^{-1}(x) = \dfrac{-5 + \sqrt{49 + 12x}}{6}}$. Our answer to (b) is $\boxed{-2 \le x}$.

(c) Take $-2 \le x$. Observe that

$$f(f^{-1}(x)) = f\left(\frac{-5 + \sqrt{49 + 12x}}{6}\right)$$

$$= 3\left(\frac{-5 + \sqrt{49 + 12x}}{6}\right)^2 + 5\left(\frac{-5 + \sqrt{49 + 12x}}{6}\right) - 2$$

$$= \frac{\left(-5+\sqrt{49+12x}\right)^2}{12} + 5\left(\frac{-5+\sqrt{49+12x}}{6}\right) - 2$$

$$= \frac{1}{12}\left[\left(-5+\sqrt{49+12x}\right)^2 + 10(-5+\sqrt{49+12x}) - 24\right]$$

$$= \frac{1}{12}\left[25 - 10\sqrt{49+12x} + 49 + 12x + 10(-5+\sqrt{49+12x}) - 24\right]$$

$$= \frac{1}{12}\left[25 + 49 + 12x - 50 - 24\right] = x. \checkmark$$

(d) Take $0 \le x$. Observe that

$$f^{-1}(f(x)) = f^{-1}(3x^2+5x-2) = \frac{-5+\sqrt{49+12(3x^2+5x-2)}}{6}$$

$$= \frac{-5+\sqrt{49+36x^2+60x-24}}{6} = \frac{-5+\sqrt{36x^2+60x+25}}{6} = \frac{-5+\sqrt{(6x+5)^2}}{6}$$

$$= \frac{-5+|6x+5|}{6}.$$

We know that $0 \le x$; so, $0 \le 6x+5$; so the most recent expression is

$$= \frac{-5+6x+5}{6} = \frac{6x}{6} = x. \checkmark$$