Recitation Time_____ PRINT your name ______________________________

Math 141, Final Exam, Solutions, Spring 2009

The exam is worth a total of 100 points. There are 20 questions on 9 pages. Each problem is worth 5 points. **SHOW your work. Make your work be coherent and clear.** Write in complete sentences whenever this is possible. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible. **No Calculators.**

I will post the solutions on my website sometime after the exam.

1. **Let** $y = e^{x\tan x}$ **. Find** $\frac{dy}{dx}$ **.**

We see that

$$\boxed{\frac{dy}{dx} = e^{x\tan x}(x\sec^2 x + \tan x).}$$

2. **Let** $y = x\ln(x^2+3x)$ **. Find** $\frac{dy}{dx}$ **.**

We see that

$$\boxed{\frac{dy}{dx} = x\frac{2x+3}{x^2+3x} + \ln(x^2+3x).}$$

3. **Let** $y = [1+\sin^3(x^5)]^{12}$ **. Find** $\frac{dy}{dx}$ **.**

We see that

$$\boxed{\frac{dy}{dx} = 12[1+\sin^3(x^5)]^{11}3\sin^2(x^5)\cos(x^5)5x^4.}$$

4. **If** $\sin(x^2y^2) = x$ **, then find** $\frac{dy}{dx}$ **.**

Take the derivative of both sides to see that

$$\cos(x^2y^2)[x^2 2y\frac{dy}{dx} + 2xy^2] = 1.$$

Now solve for $\frac{dy}{dx}$ to see that

$$\boxed{\frac{dy}{dx} = \frac{1-2xy^2\cos(x^2y^2)}{2x^2y\cos(x^2y^2)}.}$$

5. **Let** $y = \cos^2(3\sqrt{x})$ **. Find** $\frac{dy}{dx}$ **.**

We see that

$$\boxed{\frac{dy}{dx} = 2\cos(3\sqrt{x})(-\sin(3\sqrt{x}))\frac{3}{2\sqrt{x}}.}$$

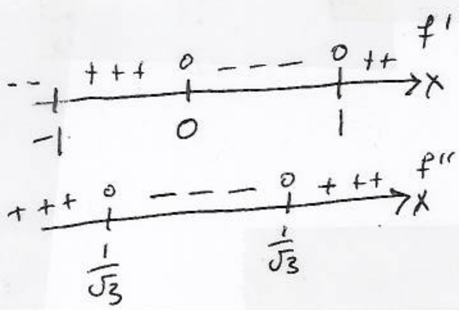
6. **Let $f(x) = x^4 - 2x^2$. Where is $f(x)$ increasing and decreasing? Where is $f(x)$ concave up and concave down? Find the local extreme points and points of inflection of $y = f(x)$. Graph $y = f(x)$.**

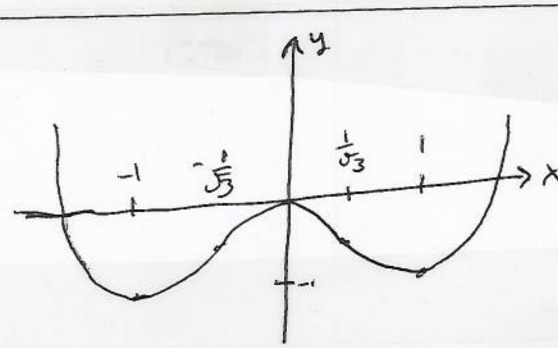
We see that $f'(x) = 4x^3 - 4x = 4x(x^2-1) = 4x(x-1)(x+1)$. The function $f(x)$ is increasing when $f'(x)$ is positive and this happens for $-1 < x < 0$, also for $1 < x$. The function $f(x)$ is decreasing when $f'(x)$ is negative and this happens for $x < -1$, also for $0 < x < 1$. The graph has a local minimum points at $(-1, f(-1))$ and $(1, f(1))$. The graph has local maximum points at $(0, f(0))$. We see that $f(-1) = -1$, $f(0) = 0$, and $f(1) = -1$.

We see that $f''(x) = 12x^2 - 4 = 4(3x^2-1) = 4(\sqrt{3}x - 1)(\sqrt{3}x + 1)$. The function $f(x)$ is concave up when $f''(x)$ is positive and this happens for $x < -1/\sqrt{3}$, also for $1/\sqrt{3} < x$. The function $f(x)$ is concave down when $f''(x)$ is negative and this happens for $-1/\sqrt{3} < x < 1/\sqrt{3}$. The points $(-1/\sqrt{3}, f(-1/\sqrt{3}))$ and $(1/\sqrt{3}, f(1/\sqrt{3}))$ are points of inflection for $y = f(x)$.

We conclude that

> $f(x)$ is increasing for $-1 < x < 0$, also for $1 < x$,
> $f(x)$ is decreasing for $x < -1$, also for $0 < x < 1$
> $f(x)$ is concave up for $x < -1/\sqrt{3}$, also for $1/\sqrt{3} < x$,
> $f(x)$ is concave down for $-1/\sqrt{3} < x < 1/\sqrt{3}$,
> $(0,0)$ is a local maximum point of $y = f(x)$,
> $(-1,-1)$ and $(1,-1)$ are local minimum points of $y = f(x)$, and
> $(-1/\sqrt{3}, f(-1/\sqrt{3}))$ and $(1/\sqrt{3}, f(1/\sqrt{3}))$ are points of inflection of $y = f(x)$.





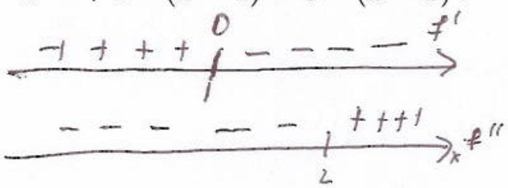
7. **Let $f(x) = xe^{-x}$. Find all vertical and horizontal asymptotes of $y = f(x)$. Where is $f(x)$ increasing and decreasing? Where is $f(x)$ concave up and concave down? Find the local extreme points and points of inflection of $y = f(x)$. Graph $y = f(x)$.**

The function $f(x)$ does not become infinite near any number x because this function never has zero in the denominator and this function never tries to take ln of zero. So there are no vertical asymptotes. It is obvious that $\lim_{x\to-\infty} f(x) = -\infty$ so there is no horizontal asymptote on the left side of the graph. However, we may use L'Hôpital's rule (since x and e^x both become infinite as x goes to ∞) to see that

$$\lim_{x\to\infty} xe^{-x} = \lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{1}{e^x} = 0;$$

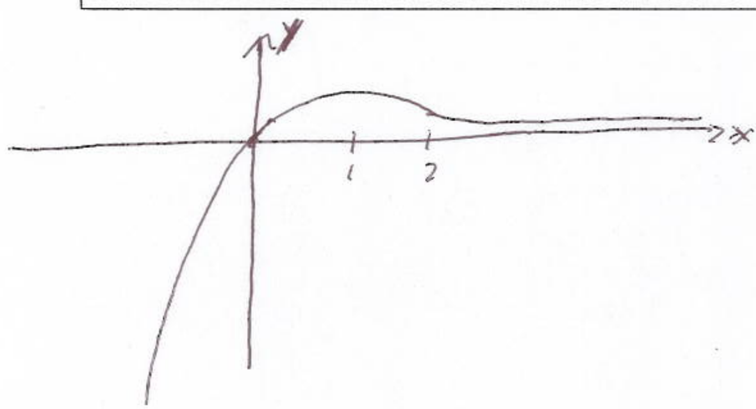
hence the x-axis is a horizontal asymptote of the graph (on the right side).

We now calculate $f'(x) = -xe^{-x} + e^{-x} = -e^{-x}(x-1)$ and
$f''(x) = -e^{-x} + e^{-x}(x-1) = e^{-x}(x-2)$.



We calculate $f(1) = e^{-1}$ and $f(2) = 2e^{-2}$. We conclude that

$f(x)$ is increasing for $x < 1$,
$f(x)$ is decreasing for $1 < x$,
$f(x)$ is concave up for $2 < x$,
$f(x)$ is concave down for $x < 2$,
$(1, e^{-1})$ is a local maximum point of $y = f(x)$,
$(2, 2e^{-2})$ is a point of inflection of $y = f(x)$, and
$y = 0$ is a horizontal asymptote of $y = f(x)$.



8. **Find $\int \frac{e^x}{1+e^x} dx$. Check your answer.**

Let $u = 1 + e^x$. It follows that $du = e^x dx$ and the problem is equal to

$$\int \frac{du}{u} = \ln|u| + C = \boxed{\ln(1 + e^x) + C.}$$

The derivative of the proposed answer is $e^x/(1 + e^x)$. ✓

9. **Find** $\int \frac{e^x}{1+e^{2x}} dx$ **. Check your answer.**

Let $u = e^x$. It follows that $du = e^x dx$ and the problem is equal to

$$\int \frac{du}{1 + u^2} = \arctan u + = \boxed{\arctan(e^x) + C}.$$

The derivative of the proposed answer is $e^x/(1 + e^{2x})$. ✓

10. **Find** $\int \frac{e^x}{\sqrt{1+e^x}} dx$ **. Check your answer.**

Let $u = 1 + e^x$. It follows that $du = e^x dx$ and the problem is equal to

$$\int u^{-1/2} du = 2\sqrt{u} + C = \boxed{2\sqrt{1 + e^x} + C.}$$

The derivative of the proposed answer is $\frac{2e^x}{2\sqrt{1+e^x}}$. ✓

11. **Find** $\int \frac{1+e^x}{e^x} dx$ **. Check your answer.**

The problem is equal to

$$\int (e^{-x} + 1) dx = \boxed{-e^{-x} + x + C}.$$

The derivative of the proposed answer is $e^{-x} + 1 = (1 + e^x)/e^x$. ✓

12. **Find** $\int e^x(1 + e^x) dx$ **. Check your answer.**

The problem is equal to

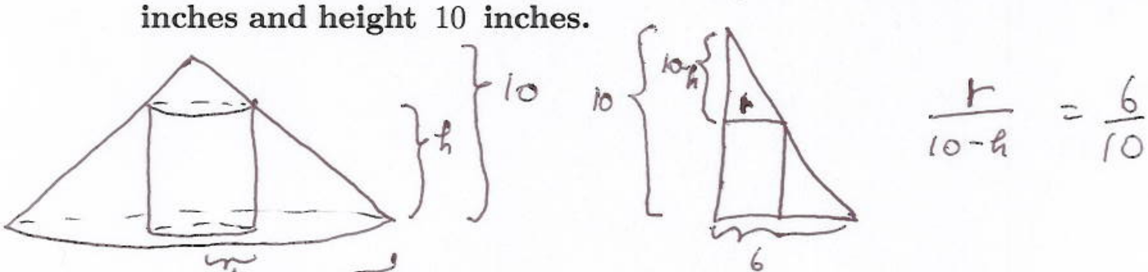
$$\int (e^x + e^{2x}) dx = \boxed{e^x + e^{2x}/2 + C}.$$

The derivative of the proposed answer is $(e^x + e^{2x})$. ✓

13. Assume that oil spilled from a ruptured tanker spreads in a circular pattern whose radius increases at a constant rate of 2 feet per second. How fast is the area of the spill increasing when the radius of the spill is 60 feet?

Let $r(t)$ be the radius of the oil spill at time t and $A(t)$ be the area of the oil spill at time t. We are told $dr/dt = 2$ feet/second. We are supposed to find dA/dt when $r = 60$ feet. But $A = \pi r^2$; so, $dA/dt = 2\pi r(dr/dt)$. When $r = 60$ feet, we have $\boxed{dA/dt = 2\pi(60)(2)\text{feet}^2/\text{second}}$.

14. Find the radius and height of the right circular cylinder of largest volume that can be inscribed in a right circular cone with radius 6 inches and height 10 inches.



Let r be the radius of the base of our cylinder, h be the height of the cylinder, and V be the volume of the cylinder. Similar triangles gives us $\frac{r}{10-h} = \frac{6}{10}$ or $\frac{5}{3}r = 10 - h$ or $h = 10 - \frac{5}{3}r$. We must maximize $V = \pi r^2 h = \pi r^2(10 - \frac{5}{3}r) = \pi(10r^2 - \frac{5}{3}r^3)$, with $0 \le r \le 6$. We see that

$$dV/dr = \pi(20r - 5r^2) = \pi 5r(4 - r).$$

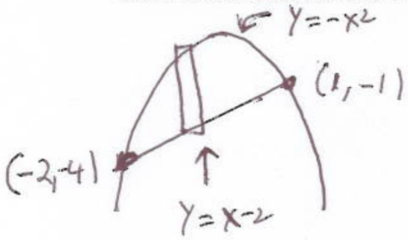
So $dV/dr = 0$ when $r = 0$ or $r = 4$. The maximum value of V occurs either at an endpoint or at a point where dV/dr is zero. We see that $V(0) = V(6) = 0$. So the maximum volume occurs when $\boxed{r = 4 \text{ inches and } h = 10/3 \text{ inches}}$.

15. An object moves in a straight line with acceleration given by $a(t) = 3\sin 3t$. If the initial velocity is $v(0) = 3$ and the initial position is $s(t) = 3$, then find the position, $s(t)$, of the object at time t.

Integrate to see that $v(t) = -\cos 3t + C$. Plug in the initial condition to see that $3 = -1 + C$ so $C = 4$ and $v(t) = -\cos 3t + 4$. Integrate again to see that

$s(t) = -\frac{\sin 3t}{3} + 4t + C_2$. Plug in the initial condition to see that $3 = C_2$. We conclude $\boxed{s(t) = -\dfrac{\sin 3t}{3} + 4t + 3}$.

16. Find the area between $y = -x^2$ and $y = x - 2$.



The intersection points occur when $-x^2 = x - 2$; so $0 = x^2 + x - 2 = 0$; or $0 = (x+2)(x-1)$; $x = -2, 1$. The area is the integral from left to right of top minus bottom:

$$\int_{-2}^{1} -x^2 - (x-2)dx = \int_{-2}^{1} (-x^2 - x + 2)dx = -\frac{x^3}{3} - \frac{x^2}{2} + 2x\Big|_{-2}^{1}$$

$$= \boxed{-1/3 - 1/2 + 2 - (8/3 - 2 - 4)}.$$

17. State either part of the Fundamental Theorem of Calculus.

Let f be a continuous function defined on the closed interval $[a, b]$.

(a) If $A(x)$ is the function $A(x) = \int_a^x f(t)dt$, for all $x \in [a, b]$, then $A'(x) = f(x)$ for all $x \in [a, b]$.

(b) If $F(x)$ is any antiderivative of $f(x)$, then $\int_a^b f(x)dx = F(b) - F(a)$.

18. Use the DEFINITION OF THE DERIVATIVE to find $f'(x)$ for $f(x) = \sqrt{2x - 1}$.

We have

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{2(x+h) - 1} - \sqrt{2x - 1}}{h}$$

$$= \lim_{h \to 0} \frac{(\sqrt{2(x+h) - 1} - \sqrt{2x - 1})(\sqrt{2(x+h) - 1} + \sqrt{2x - 1})}{h(\sqrt{2(x+h) - 1} + \sqrt{2x - 1})}$$

$$= \lim_{h \to 0} \frac{2(x+h) - 1 - (2x - 1)}{h(\sqrt{2(x+h) - 1} + \sqrt{2x - 1})} = \lim_{h \to 0} \frac{2h}{h(\sqrt{2(x+h) - 1} + \sqrt{2x - 1})}$$

$$= \lim_{h\to 0} \frac{2}{(\sqrt{2(x+h)-1}+\sqrt{2x-1})} = \frac{2}{(\sqrt{2x-1}+\sqrt{2x-1})} = \frac{2}{2\sqrt{2x-1}}$$

$$= \frac{1}{\sqrt{2x-1}}.$$

19. **Compute** $\lim_{x\to 0^+}(1-2x)^{1/3x}$ **.**

Let $y=(1-2x)^{1/3x}$. We compute

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{\ln(1-2x)}{3x}.$$

The top and the bottom of the most recent expression both go to 0 . We apply L'Hopital's rule to see that

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{\frac{-2}{1-2x}}{3} = \frac{-2}{3}.$$

Thus the answer is

$$\lim_{x\to 0^+} y = \lim_{x\to 0^+} e^{\ln y} = \boxed{e^{-2/3}}.$$

20. **Find the equation of the line tangent to** $f(x)=3x^2+5x$ **at** $x=1$ **.**

We calculate $f(1)=3+5=8$, $f'(x)=6x+5$, and $f'(1)=6+5=11$. The line through $(1,8)$ with slope 11 is

$$\boxed{y-8=11(x-1)}.$$