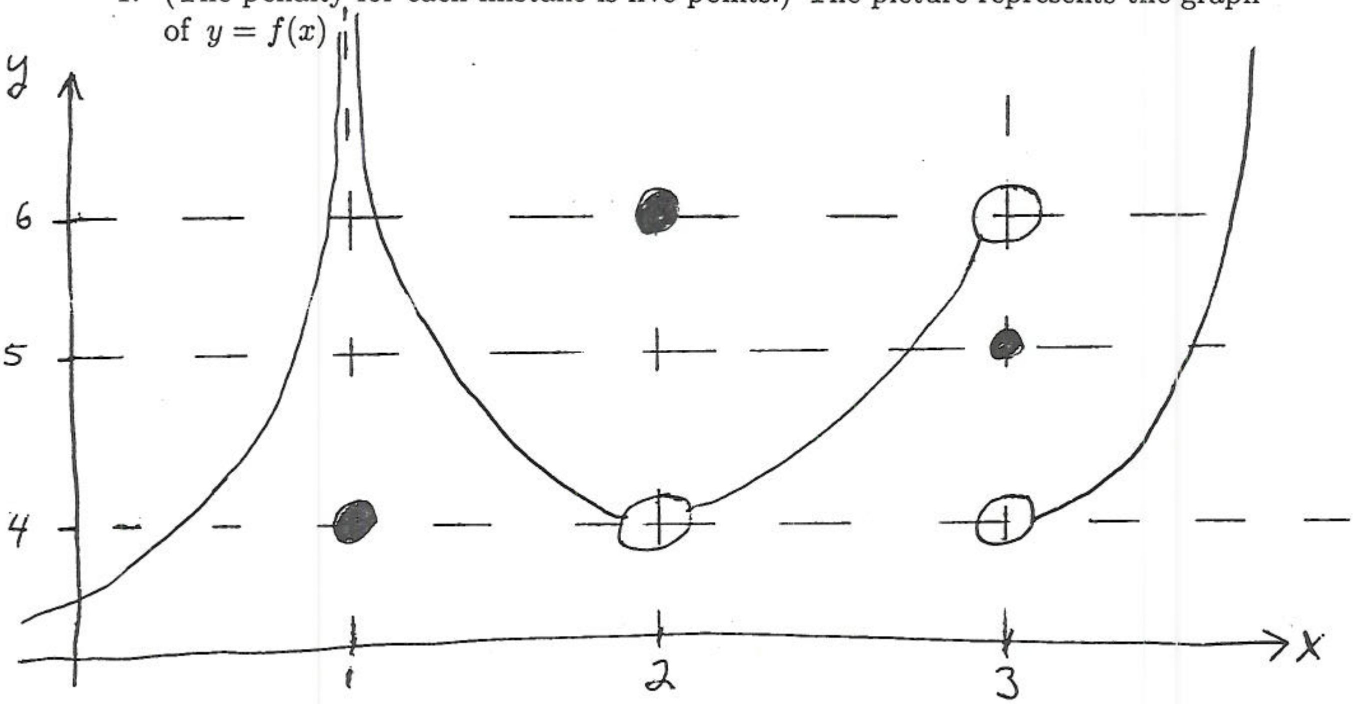
Final Exam 141 2000

PRINT Your Name:____________________________

There are 19 problems on 9 pages. Problem 1 is worth 20 points, each of the other problems is worth 10 points. SHOW your work. $\boxed{\textit{CIRCLE}}$ your answer. **NO CALCULATORS!**

1. (The penalty for each mistake is five points.) The picture represents the graph of $y = f(x)$



(a) Fill in the blanks:

$f(1) = 4$	$\lim_{x\to1^+} f(x) = +\infty$	$\lim_{x\to1^-} f(x) = +\infty$	$\lim_{x\to1} f(x) = +\infty$
$f(2) = 6$	$\lim_{x\to2^+} f(x) = 4$	$\lim_{x\to2^-} f(x) = 4$	$\lim_{x\to2} f(x) = 4$
$f(3) = 5$	$\lim_{x\to3^+} f(x) = 4$	$\lim_{x\to3^-} f(x) = 6$	$\lim_{x\to3} f(x) =$ DNE

(b) Where is f discontinuous? at $x = 1, 2, 3$

(c) Where is f not differentiable? at $x = 1, 2, 3$

2. What is the derivative of $f(x) = 2x^5 + \frac{3}{\sqrt{x}} + 4x^2 + 10$?

$$f'(x) = 10x^4 - \tfrac{3}{2}x^{-\frac{3}{2}} + 8x$$

3. Use the DEFINITION of the DERIVATIVE to find the derivative of $f(x) = x^2$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}} = \lim_{h \to 0} 2x + h = 2x.$$

4. What is the equation of the line tangent to $f(x) = 2x^3 - 2$ at the point where $x = 3$.

$f'(x) = 6x^2$

$f'(3) = 6(3)^2 = 54$

$f(3) = 2(3^3) - 2 = 2(27) - 2 = 52$

$y - 52 = 54(x - 3)$

5. If $y = \frac{(5x^3+9x)^2}{\sin(7x^2-15x)}$, then find $\frac{dy}{dx}$.

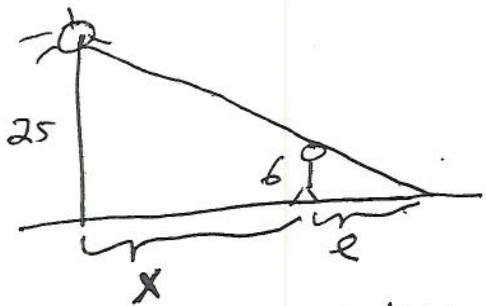
$$\frac{dy}{dx} = \frac{\sin(7x^2-15x)\, 2(5x^3+9x)(15x^2+9) - (5x^3+9x)^2 \cos(7x^2-15x)(14x-15)}{\sin^2(7x^2-15x)}$$

6. Find $\frac{dy}{dx}$ for $6x^2y^2 + 2x = \cos y$.

$$6x^2\, 2y \frac{dy}{dx} + 12xy^2 + 2 = -\sin y \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{-2-12xy^2}{12x^2y+\sin y}$$

7. A man, who is 6 feet tall, is walking, at the rate of 3 ft./sec., away from a light pole, which is 25 feet high. How fast is his shadow growing when he is 30 feet from the light pole?



We see $\frac{dl}{dt} = \frac{6}{19}\frac{dx}{dt} = \frac{6}{19}(3) = \frac{18}{19}$ ft/s

x = dist from man to light
l = length of shadow

$$\frac{l}{6} = \frac{x+l}{25}$$

$$25l = 6(x+l)$$

$$19l = 6x$$

$$19\frac{dl}{dt} = 6\frac{dx}{dt}$$

we know $\frac{dx}{dt} = 3$ ft/s

8. DEFINE the definite integral $\int_a^b f(x)\,dx$.

Let $f(x)$ be a function defined for $a \le x \le b$. For each partition P of the interval $a \le x \le b$ of the form $a = x_0 \le x_1 \le \dots \le x_n = b$, let $U_P(f) = \sum_{i=1}^{n} M_i (x_i - x_{i-1})$ and $L_P(f) = \sum_{i=1}^{n} m_i (x_i - x_{i-1})$, where M_i is the maximum of $f(x)$ for $x_{i-1} \le x \le x_i$ and m_i is the minimum of $f(x)$ for $x_{i-1} \le x \le x_i$. If there is exactly one number between every $L_P(f)$ and every $U_P(f)$, then that number is the definite integral of $f(x)$ on the interval $a \le x \le b$ and that number is denoted $\int_a^b f(x)\,dx$.

9. Find $\int (2x^4 + \sqrt{3x-2}\,)\,dx$. Check your answer.

$$\frac{2x^5}{5} + \frac{2}{9}(3x-2)^{\frac{3}{2}} + C$$

10. Find $\int x\cos(5x^2+18)\,dx$. Check your answer.

$$\frac{1}{10}\sin(5x^2+18) + C$$

11. Find $\int_0^1 5x\sqrt{3x^2+4}dx$. Check your answer.

$$= \frac{10}{3}\cdot\frac{1}{6}(3x^2+4)^{\frac{3}{2}}\Big]_0^1 = \frac{10}{18}\left((7)^{\frac{3}{2}} - 4^{\frac{3}{2}}\right)$$

12. Let $f(x) = x^3 - 3x$. Where is $f(x)$ increasing, decreasing, concave up, and concave down? Find the local maximum points, local minimum points and the points of inflection of $y = f(x)$. Find the vertical and horizontal asymptotes of $y = f(x)$. GRAPH $y = f(x)$.

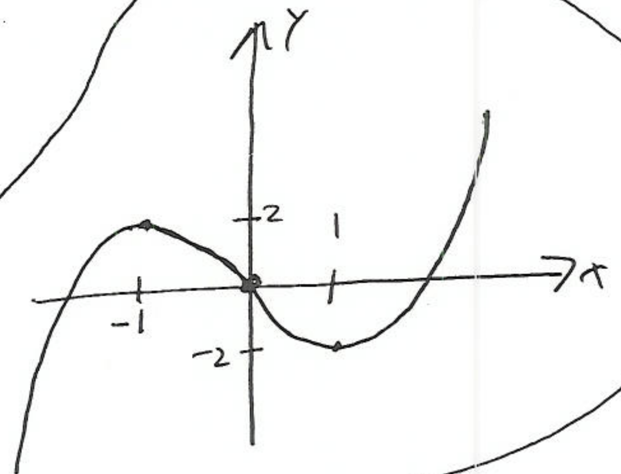
No Horizontal or Vertical Asymptotes

$f'(x) = 3x^2 - 3 = 3(x-1)(x+1)$

$f''(x) = 6x$

f': $+$ $\quad -1 \quad$ $-$ $\quad 1 \quad$ $+$

f'': $-$ $\quad 0 \quad$ $+$



f is inc. for $x < -1$ also for $1 < x$

f is dec. for $-1 < x < 1$

f is c.u. for $0 < x$

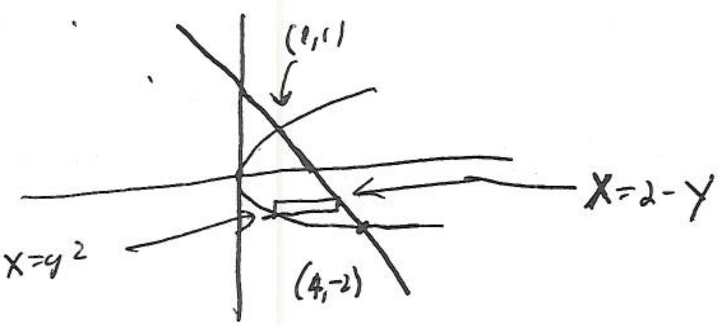
f is c.d. for $x < 0$

$(1, -2)$ loc min

$(-1, 2)$ loc max

$(0, 0)$ p of i

13. Find the area between $x = y^2$ and $x + y = 2$.



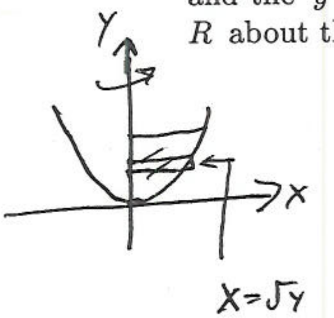
intersection: $y^2 + y - 2 = 0$

$(y+2)(y-1) = 0$

$y = -2, 1$

$$\text{area} = \int_{-2}^{1} 2 - y - y^2 \, dy = 2y - \frac{y^2}{2} - \frac{y^3}{3} \Big]_{-2}^{1} = \left(2 - \frac{1}{2} - \frac{1}{3} - \left(-4 - 2 + \frac{8}{3}\right)\right)$$

14. Let R be the region in the first quadrant which is bounded by $y = x^2$, $y = 1$, and the $y-$ axis. Find the volume of the solid which is obtained by revolving R about the $y-$ axis.



Spin the rect. Get a disc
of vol $\pi r^2 t$ where
$t = dy$, $r = \sqrt{y}$

Total vol $= \pi \int_0^1 y\,dy = \frac{\pi y^2}{2}\Big]_0^1 = \frac{\pi}{2}$

15. Find the length of $y = (4 - x^{2/3})^{3/2}$ from $x = 1$ to $x = 8$.

$AL = \int_1^8 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$

$= \int_1^8 \sqrt{1 + \left(\frac{3}{2}(4 - x^{\frac{2}{3}})^{\frac{1}{2}}\left(-\frac{2}{3}x^{-\frac{1}{3}}\right)\right)^2}\,dx$

$= \int_1^8 \sqrt{1 + (4 - x^{\frac{2}{3}})(x^{-\frac{2}{3}})}\,dx$

$= \int_1^8 \sqrt{4x^{-\frac{2}{3}}}\,dx$

$= \int_1^8 2x^{-\frac{1}{3}}\,dx = 2\left(\frac{3}{2}\right)x^{\frac{2}{3}}\Big]_1^8 = 3\left(8^{\frac{2}{3}} - 1\right)$

16. Solve the initial value problem $\frac{dy}{dx} = y^5$, $y(2) = 1$. Check your answer.

$\int y^{-5} dy = \int dx$

$\frac{y^{-4}}{-4} = x + C$

$\frac{1}{y^4} = -4x - 4C$

when $x=2, y=1$ so

$1 = -8 - 4C$

$9 = -4C$

so $\frac{1}{y^4} = 9 - 4x$

$\frac{1}{9-4x} = y^4$

$\frac{1}{(9-4x)^{\frac{1}{4}}} = y$

Check

when $x=2$ $y = \frac{1}{(9-8)^{\frac{1}{4}}} = 1$ ✓

$\frac{dy}{dx} = \cancel{-\frac{1}{4}}(9-4x)^{-\frac{5}{4}}(\cancel{-4})$

$y^5 = \frac{1}{(9-4x)^{\frac{5}{4}}}$

equal!

17. Let $f(x) = 16x^{\frac{1}{3}} + x^{\frac{4}{3}}$. Where is $f(x)$ increasing, decreasing, concave up, and concave down? Find the local maximum points, local minimum points and the points of inflection of $y = f(x)$. Find the vertical and horizontal asymptotes of $y = f(x)$. GRAPH $y = f(x)$.

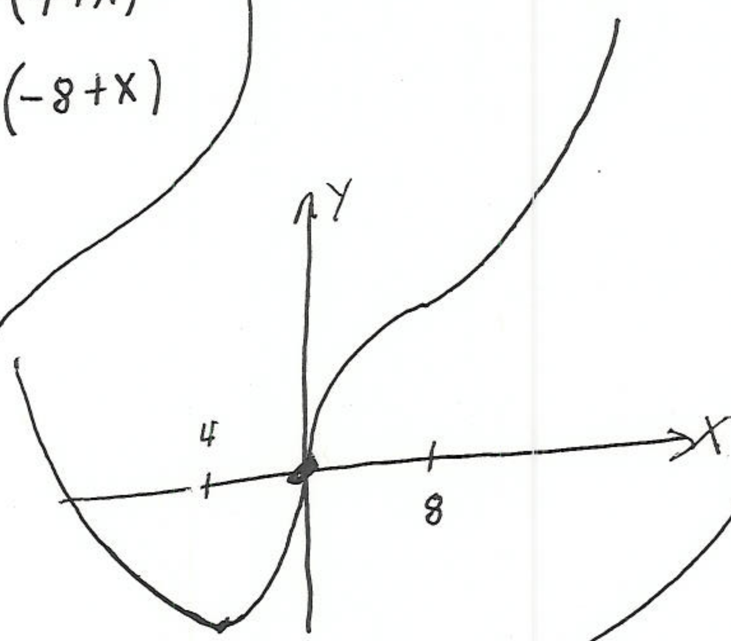
No Vertical or Horizontal asymptotes.

$f'(x) = 16\left(\frac{1}{3}\right)x^{-\frac{2}{3}} + \frac{4}{3}x^{\frac{1}{3}} = \frac{4}{3}x^{-\frac{2}{3}}(4+x)$

$f''(x) = -\frac{32}{9}x^{-\frac{5}{3}} + \frac{4}{9}x^{-\frac{2}{3}} = \frac{4}{9}x^{-\frac{5}{3}}(-8+x)$

− | + | +
−4, 0

+ | − | +
0, 8



f is inc for $-4<x$

f is dec for $x<4$

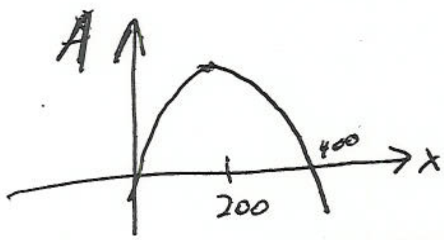
f is c.u. for $8<x$ also for $x<0$

f is c.d. for $0<x<8$

loc. min $(-4, f(-4))$

pts i $(0,0)$ · $(8, f(8))$

18. A farmer has 800 feet of fencing with which he plans to enclose a rectangular pen adjacent to a long existing wall. He will use the wall for one side of the pen and the available fencing for the remaining three sides. What is the maximum possible area that he can enclose in this way?



$A = xy$

$800 = 2x + y$

$y = 800 - 2x$

$A = x(800 - 2x)$

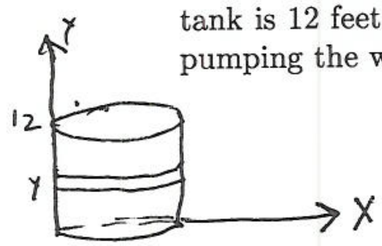
$A = 800x - 2x^2$

$A' = 800 - 4x$

$A' = 0$ when $x = 200$

The area is maximum when the pen is 200 ft by 400 feet

19. A tank in the shape of a right circular cylinder, standing on its end, is full of water. The density of water is 62.4 pounds per cubic foot. If the height of the tank is 12 feet and the radius of the tank is 6 feet, then find the work done in pumping the water over the top edge of the tank.



The work to lift the layer of H_2O at y is

force · distance

= wt · distance

$= \pi r^2 t\,(12-y)\,62.4$

$= \pi\, 36\,(12-y)\,dy\; 62.4$

Total work $= 36\pi \int_0^{12} (12-y)\,dy\,(62.4)$

$= 36\pi \left(12y - \frac{y^2}{2}\right)\Big]_0^{12} (62.4)$

$= 36\pi \left(144 - \frac{144}{2}\right)(62.4) - 165$