1996 Final Exam

S7

PRINT Your Name:______________________ Section:__________

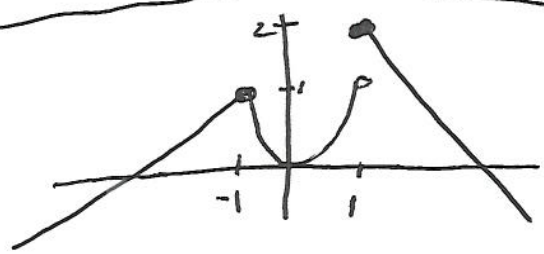
There are 19 problems on 8 pages. Problem 1 is worth 20 points. Each of the other problem is worth 10 points. SHOW your work. $\boxed{CIRCLE}$ your answer. **NO CALCULATORS!** You might find the following formulas to be useful:

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \quad \text{and} \quad \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}.$$

1. Let

$$f(x) = \begin{cases} x+2 & \text{if } x \le -1, \\ x^2 & \text{if } -1 < x < 1, \\ -x+3 & \text{if } 1 \le x. \end{cases}$$

(a) Graph $y = f(x)$.




(b) Fill in the blanks:

| | | | |
|---|---|---|---|
| $f(-1) = \underline{1}$ | $\lim_{x\to -1+} f(x) = \underline{1}$ | $\lim_{x\to -1-} f(x) = \underline{1}$ | $\lim_{x\to -1} f(x) = \underline{1}$ |
| $f(0) = \underline{0}$ | $\lim_{x\to 0+} f(x) = \underline{0}$ | $\lim_{x\to 0-} f(x) = \underline{0}$ | $\lim_{x\to 0} f(x) = \underline{0}$ |
| $f(1) = \underline{2}$ | $\lim_{x\to 1+} f(x) = \underline{2}$ | $\lim_{x\to 1-} f(x) = \underline{1}$ | $\lim_{x\to 1} f(x) = \underline{DNE}$ |

2. Use the DEFINITION of the DERIVATIVE to find the derivative of $f(x) = \sqrt{3x-5}$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{3(x+h)-5} - \sqrt{3x-5}}{h}$$

$$= \lim_{h\to 0} \frac{3(x+h)-5-(3x-5)}{h\left(\sqrt{3(x+h)-5} + \sqrt{3x-5}\right)} = \lim_{h\to 0} \frac{3}{\sqrt{3(x+h)-5} + \sqrt{3x-5}}$$

$$= \frac{3}{2\sqrt{3x-5}}$$

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3. State both parts of the Fundamental Theorem of Calculus.

Let $f(x)$ be a continuous function.

1) If $A(x) = \int_a^x f(t)dt$, then $A'(x) = f(x)$

2) If $F'(x) = f(x)$, then $\int_a^b f(t)dt = F(b) - F(a)$

4. Define the definite integral $\int_a^b f(x)dx$.

For each partition $P$ of the interval $[a,b]$ of the form
$a = x_0 \le x_1 \le \dots \le x_{n-1} \le x_n = b$, let
$m_i$ be the minimum value of $f(x)$ on $x_{i-1} \le x \le x_i$,
$M_i$ be the maximum value of $f(x)$ on $x_{i-1} \le x \le x_i$,
$L_p(f) = \sum_{i=1}^{n} m_i (x_i - x_{i-1})$, and $U_p(f) = \sum_{i=1}^{n} M_i (x_i - x_{i-1})$.

If there is exactly one number between every $L_p(f)$ and every $U_p(f)$, then that number is called the definite integral of $f(x)$ from $x=a$ to $x=b$ and we write it as $\int_a^b f(x)\,dx$

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5. Let $f(x) = \dfrac{4x}{x^2+2}$. Where is $f(x)$ increasing, decreasing, concave up, and concave down? What are the local maximum points, local minimum points, and points of inflection of $y = f(x)$. Find all vertical and horizontal asymptotes of $y = f(x)$. Graph $y = f(x)$.

$\lim_{x\to\infty} f(x) = 0$ $\lim_{x\to-\infty} f(x) = 0$ so the x-axis is a horiz. asy.

The denominator is never zero so there are no vertical asymptotes

$$f' = \frac{(x^2+2)4 - 4x(2x)}{(x^2+2)^2} = (8-4x^2)(x^2+2)^{-2}$$

$f' = 0$ when $x = \sqrt{2}$ or $-\sqrt{2}$

$$f'' = (8-4x^2)(-2)(x^2+2)^{-3}2x + (x^2+2)^{-2}(-8x)$$

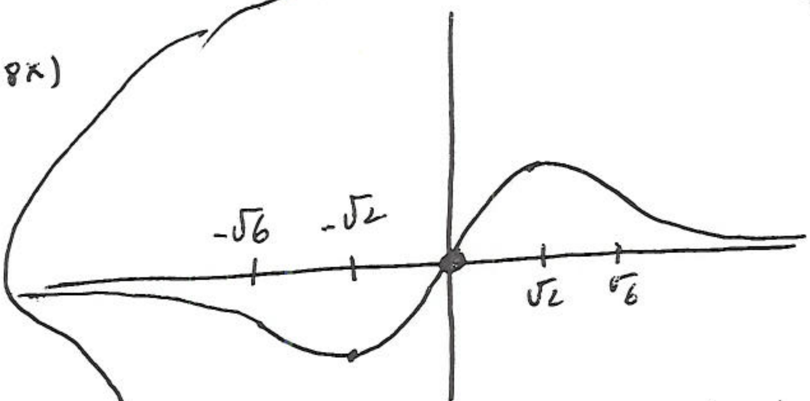
$$= \frac{-8x(\cancel{4}\,2x^2 + x^2+2)}{(x^2+2)^3}$$

$$= \frac{-8x(6-x^2)}{(x^2+2)^3}$$

$f'' = 0$ when $x = 0, \sqrt{6}$, or $-\sqrt{6}$

f' neg | $-\sqrt{2}$ | f' pos | $\sqrt{2}$ | f' neg

− | $-\sqrt{6}$ | f'' pos | 0 | f'' Neg | $\sqrt{6}$ | f'' pos




P of i $(-\sqrt{6}, f(-\sqrt{6}))$ $(0,0)$ $(\sqrt{6}, f(\sqrt{6}))$

loc min $(-\sqrt{2}, f(-\sqrt{2}))$

loc max $(\sqrt{2}, f(\sqrt{2}))$

inc $-\sqrt{2} < x < \sqrt{2}$

dec $x < -\sqrt{2}$ or $\sqrt{2} < x$

c.d. $x < -\sqrt{6}$ or $0 < x < \sqrt{6}$

c.u. $-\sqrt{6} < x < 0$ or $\sqrt{6} < x$

6. Find $\lim_{x\to3^+} \dfrac{x-3}{x^2-2x+3}$. $= \dfrac{0}{9-6+3} = \dfrac{0}{6} = 0$

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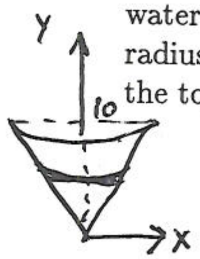
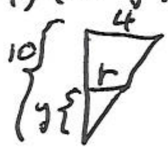
7. A tank in the shape of a right circular cone is full of water. The density of water is 62.4 pounds per cubic foot. If the height of the tank is 10 feet and the radius of its top is 4 feet, then find the work done in pumping the water over the top edge of the tank.




The work to lift the layer of water of thickness $dy$ at $y$-coordinate $y$ is wt · dist $= 62.4 \cdot \text{vol} \cdot \text{dist}$

$= (62.4)(10-y)\,\pi r^2\,dy$

where 

$\frac{r}{y} = \frac{4}{10}$ $r = \frac{2}{5}y$

$= (62.4)(10-y)\frac{\pi 4}{25}y^2\,dy$

$$\text{Work} = \int_0^{10} (62.4)(10-y)\frac{4}{25}y^2\,dy$$

$$= (62.4)\frac{4\pi}{25}\int_0^{10} 10y^2 - y^3\,dy$$

$$(62.4)\frac{4\pi}{25}\left[\frac{10y^3}{3} - \frac{y^4}{4}\right]_0^{10}$$

$$\boxed{= (62.4)\frac{4\pi}{25}\left(\frac{(10)^4}{3} - \frac{10^4}{4}\right) \text{ ft-lbs}}$$

8. Let $f(x) = \sqrt{\sin^3 2x + \frac{3}{x}}$. Find $f'(x)$.

$$\boxed{f' = \frac{3\sin^2 2x(\cos 2x)2 - \frac{3}{x^2}}{2\sqrt{\sin^3 2x + \frac{3}{x}}}}$$

9. Find the equation of the line tangent to $y = 3x^{10}$ at $x = 1$.

$y' = 30x^9$ $y'(1) = 30$ $y(1) = 3$

$$\boxed{y - 3 = 30(x-1)}$$

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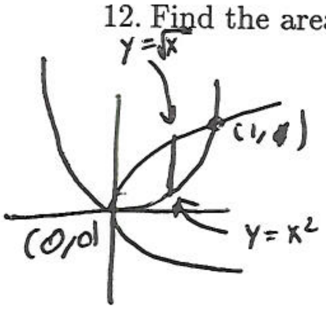
10. Find $\frac{dy}{dx}$ for $3x^2y^3 + \cos(xy) = 19y$.

$3x^2 3y^2 y' + 6xy^3 - \sin(xy)[xy' + y] = 19y'$

$$\boxed{\frac{6xy^3 - y\sin(xy)}{19 - 9x^2y^2 + x\sin(xy)} = y'}$$

11. Compute $\displaystyle\int \frac{\sin 2x}{\sqrt{1+\cos 2x}}\,dx$. $= -\sqrt{1+\cos 2x} + C$

12. Find the area of the region which is bounded by $y = x^2$ and $x = y^2$.




$\displaystyle\int_0^1 \sqrt{x} - x^2\,dx = \frac{2}{3}x^{\frac{3}{2}} - \frac{x^3}{3}\Big]_0^1$

$= \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$

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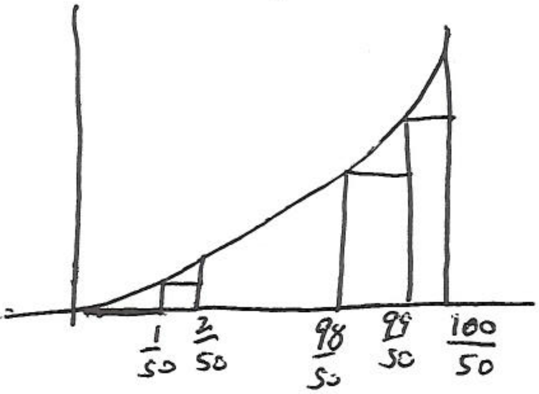
13. Consider the region $A$, which is bounded by the $x-$ axis, $y = x^2$, $x = 0$, and $x = 2$. Consider 100 rectangles, all with base $1/50$, which UNDER estimate the area of $A$. How much area is inside the 100 rectangles? (You must answer the question I asked. I expect an exact answer in closed form.)




The area inside the boxes $= \sum_{k=0}^{99} \frac{1}{50}\left(\frac{k}{50}\right)^2$

$$= \frac{1}{50^3} \sum_{k=1}^{99} k^2$$

$$= \boxed{\frac{1}{50^3} \cdot \frac{99(100)(199)}{6}}$$

14. Solve the Initial Value Problem $\frac{dy}{dx} = y^3$, $y(1) = 1$.

$$\int \frac{dy}{y^3} = \int dx$$

$$\frac{-1}{2y^2} = x + C$$

$$-\frac{1}{2} = 1 + C$$

$$-\frac{3}{2} = C$$

$$\frac{1}{2y^2} = \frac{3}{2} - x$$

$$\frac{1}{3-2x} = y^2$$

$$\frac{1}{\sqrt{3-2x}} = y$$

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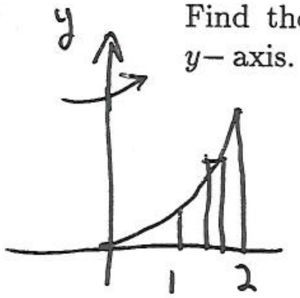
15. Let $R$ be the region between $y = x^2$ and the $x-$axis, from $x = 1$ to $x = 2$. Find the volume of the solid which is obtained by revolving $R$ about the $y-$axis.




shells $2\pi r h t$

$t = dx \quad h = x^2 \quad r = x$

$$2\pi \int_1^2 x x^2\,dx = 2\pi \int_1^2 x^3\,dx = 2\pi \left.\frac{x^4}{4}\right]_1^2$$

$$= 2\pi\left(1 - \frac{1}{4}\right)$$

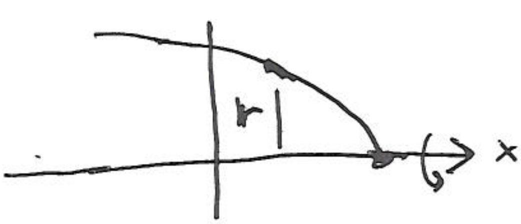
16. Find the length of the curve $y = \int_1^x \sqrt{t^3 - 1}\,dt$, from $x = 1$ to $x = 2$.

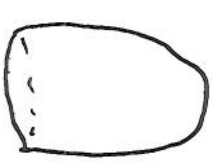
$$\int_1^2 \sqrt{\left(\frac{dy}{dx}\right)^2 + 1}\,dx = \int_1^2 \sqrt{\left(\sqrt{x^3 - 1}\right)^2 + 1}\,dx = \int_1^2 \sqrt{x^3 - 1 + 1}\,dx$$

FTC

$$= \int_1^2 x^{\frac{3}{2}}\,dx = \left.\frac{2}{5}x^{\frac{5}{2}}\right]_1^2 = \frac{2}{5}\left[(2)^{\frac{5}{2}} - 1\right]$$

17. Find the area of the surface which is generated by revolving $x = 1 - y^2$, for $0 \le y \le 2$, about the $x-$axis.



$$\int_0^2 2\pi r\,ds = \int_0^2 2\pi y \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\,dy$$

$$= \int_0^2 2\pi y \sqrt{(-2y)^2 + 1}\,dy$$

$$= \left. 2\pi \left(\frac{1}{8}\right)\frac{2}{3}\left(4y^2 + 1\right)^{\frac{3}{2}}\right]_0^2$$

$$= \frac{\pi}{6}\left((17)^{\frac{3}{2}} - 1\right)$$

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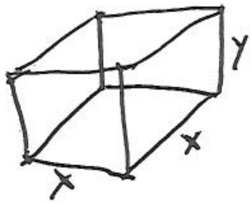
18. A rectangular box has a square base and no top. If the total area of its five sides is 108 square inches, then what is the maximum possible volume of such a box?

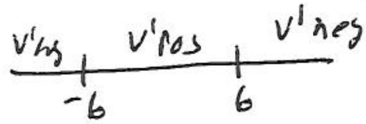


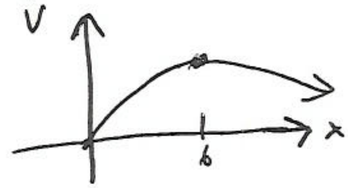

$x^2 + 4xy = 108 \qquad y = \frac{108 - x^2}{4x} = \frac{27}{x} - \frac{x}{4}$

$V = x^2 y$

$V = x^2\left(\frac{27}{x} - \frac{x}{4}\right) = 27x - \frac{x^3}{4}$

$V' = 27 - \frac{3x^2}{4} = \frac{3}{4}(36 - x^2) = \frac{3}{4}(6-x)(6+x)$




to maximize volume take $x = 6$ in and $y = 3$ in

the volume then is 108 in$^3$

19. Each edge of a cube is increasing at the rate of 4 inches per second. How fast is the volume of the cube increasing when an edge is 12 inches long?

$V = s^3$

$\frac{dV}{dt} = 3s^2 \frac{ds}{dt}$

$$\boxed{\left.\frac{dV}{dt}\right|_{s=12} = 3(144)\,4 \ \frac{\text{in}^3}{\text{sec.}}}$$

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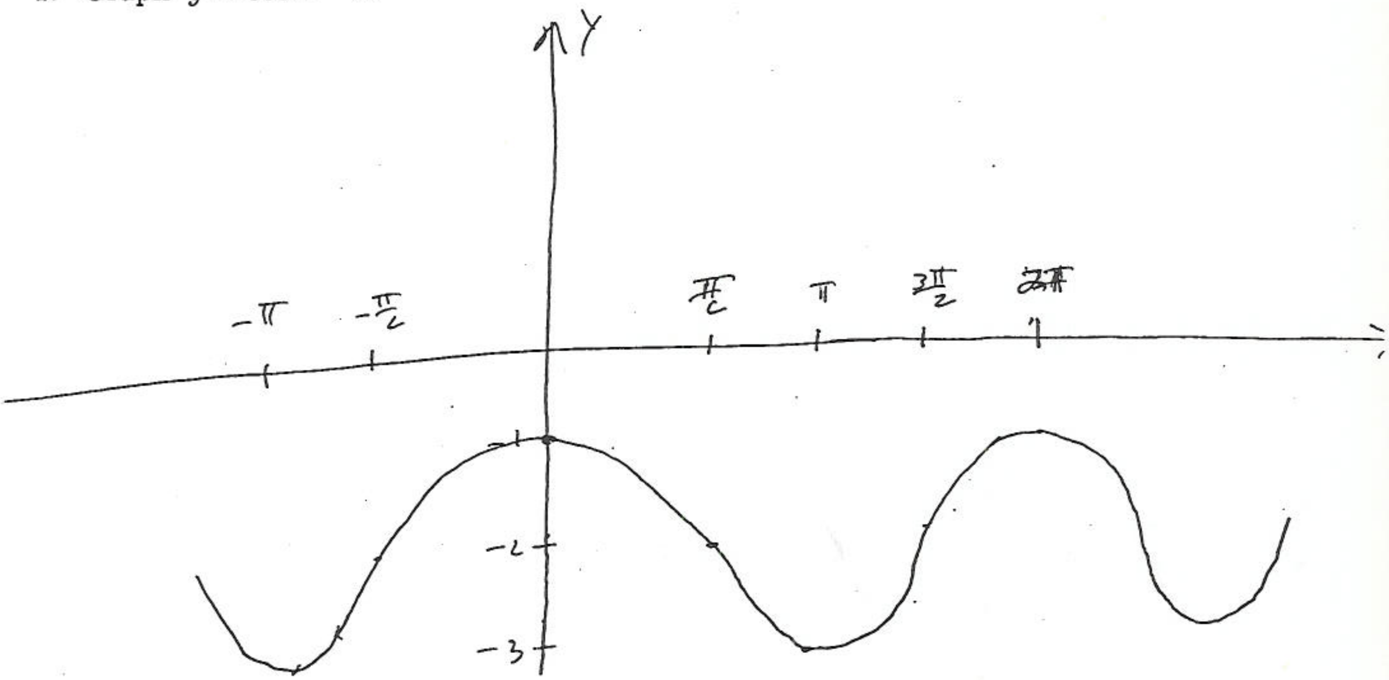
PRINT Your Name:______________________________ Section:__________

There are 9 problems on 4 pages. Each problem, unless otherwise noted, is worth 10 points. In one problem you are instructed to use the definition of the derivative; you MUST use the definition of the derivative in that problem. In the other problems you may use any legitimate derivative rule. SHOW your work. $\boxed{CIRCLE}$ your answer. **NO CALCULATORS!**

1. Graph $y = \cos x - 2$.




2. Let $f(x) = 9x^4 + \dfrac{8}{x} + 3\sqrt{x} + 6$. Find $f'(x)$.

$f(x) = 9x^4 + 8x^{-1} + 3x^{\frac{1}{2}} + 6$

$f'(x) = 36x^3 - 8x^{-2} + \frac{3}{2}x^{-\frac{1}{2}}$