1995 Final Exam

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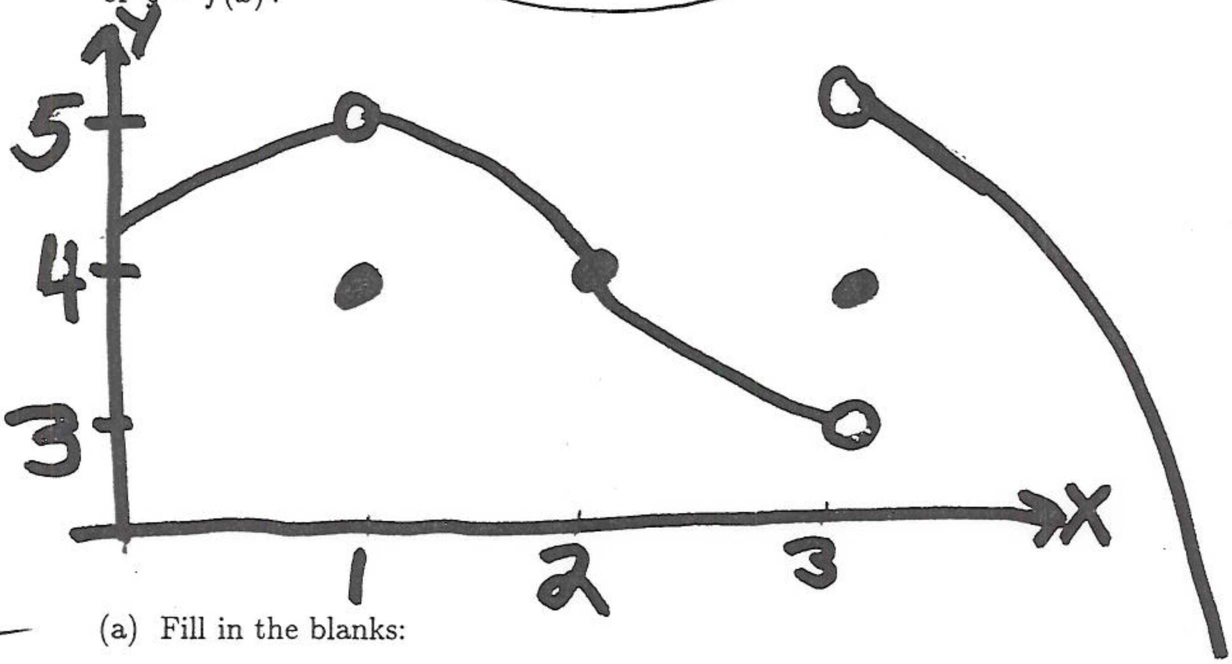
PRINT Your Name: Key There are 19 problems on 10 pages. The exam is worth 200 points. Problems 1 and 3 are each worth 15 points. Each of the other problems is worth 10 points. SHOW your work. CIRCLE your answer. **NO CALCULATORS!!!**

1. (The penalty for each mistake is five points.) The picture represents the graph of $y = f(x)$.



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(a) Fill in the blanks:

$f(1) = 4$ $\lim_{x\to1^+} f(x) = 5$ $\lim_{x\to1^-} f(x) = 5$ $\lim_{x\to1} f(x) = 5$

$f(2) = 4$ $\lim_{x\to2^+} f(x) = 4$ $\lim_{x\to2^-} f(x) = 4$ $\lim_{x\to2} f(x) = 4$

$f(3) = 4$ $\lim_{x\to3^+} f(x) = 5$ $\lim_{x\to3^-} f(x) = 3$ $\lim_{x\to3} f(x) =$ DNE

(b) Where is f discontinuous? $x = 1, 3$

(c) Where is f not differentiable? $x = 1, 3$

2. What is the equation of the line tangent to $f(x) = 2x^9 - 3x^2$ at the point where $x = 1$.

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$f(1) = -1$

$f'(x) = 18x^8 - 6x$

$f'(1) = 12$

$y + 1 = 12(x - 1)$

$\boxed{y = 12x - 13}$

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3. (The penalty for each mistake is five points.) Let

$$f(x) = \begin{cases} x+1 & \text{if } x \le 1, \\ x^2-1 & \text{if } 1 < x < 2, \\ -x+5 & \text{if } 2 \le x. \end{cases}$$

(a) Graph $y = f(x)$.

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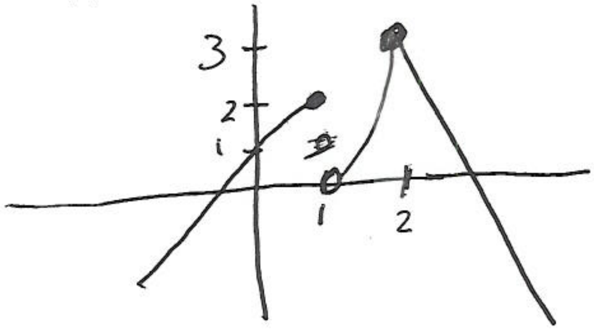

(b) Fill in the blanks:

$f(1) = 2$ $\quad \lim_{x\to1^+} f(x) = 0$ $\quad \lim_{x\to1^-} f(x) = 2$ $\quad \lim_{x\to1} f(x) =$ DNE

$f(2) = 3$ $\quad \lim_{x\to2^+} f(x) = 3$ $\quad \lim_{x\to2^-} f(x) = 3$ $\quad \lim_{x\to2} f(x) = 3$

$f(3) = 2$ $\quad \lim_{x\to3^+} f(x) = 2$ $\quad \lim_{x\to3^-} f(x) = 2$ $\quad \lim_{x\to3} f(x) = 2$

(c) Where is f discontinuous? $x = 1$

(d) Where is f not differentiable? $x = 1, 2$

4. Use the DEFINITION of the DERIVATIVE to find the derivative of $f(x) = \sqrt{2x-1}$.

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$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sqrt{2(x+\Delta x)-1} - \sqrt{2x-1}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2(x+\Delta x)-1 - (2x-1)}{\left(\sqrt{2(x+\Delta x)-1} + \sqrt{2x-1}\right)\Delta x} = \lim_{\Delta x \to 0} \frac{2\cancel{\Delta x}}{\cancel{\Delta x}\left(\sqrt{2(x+\Delta x)-1} + \sqrt{2x-1}\right)}$$

$$= \frac{2}{2\sqrt{2x-1}} = \boxed{\frac{1}{\sqrt{2x-1}}}$$

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5. If $y = \dfrac{\sin(7x^2 + 3x^2 - 15x)}{(4x^5 + 5x^3 + 9x)^2}$, then find $\frac{dy}{dx}$.

$$y' = \frac{(4x^5+5x^3+9x)^2 \cos(7x^2+3x^2-15x)(14x+6x-15) - \sin(10x^2-15x)\,2(4x^5+5x^3+9x)((20x^4+15x^2+9)}{(4x^5+5x^3+9x)^4}$$

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6. Find $\frac{dy}{dx}$ for $6x^3y^2 + 2x = x\cos y$.

$$12x^3y\,y' + 18x^2y^2 + 2 = -x\sin y\,y' + \cos y$$

$$\boxed{y' = \frac{\cos y - 18x^2y^2 - 2}{12x^3y + x\sin y}}$$

10

7. STATE both parts of the Fundamental Theorem of Calculus.

Let f be cont. for $a \le x \le b$

1) If $A(x) = \int_a^x f(t)dt$, then $A'(x) = f(x)$

2) If $F'(x) = f(x)$, then $\int_a^b f(t)\,dt = F(b) - F(a)$

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8. DEFINE the definite integral $\int_a^b f(x)\,dx$.

For each partition $P: a = x_0 \leq x_1 \leq \dots \leq x_n = b$, let

10 $U_P = M_1(x_1 - x_0) + \dots + M_n(x_n - x_{n-1})$

$L_P = m_1(x_1 - x_0) + \dots + m_n(x_n - x_{n-1})$

when M_i = max f on $[x_{i-1}, x_i]$

and m_i = min f on $[x_{i-1}, x_i]$.

If there is exactly one # with $L_P \leq \# \leq U_P$ for all P, then that number is $\int_a^b f(x)\,dx$.

9. A 30-foot ladder is leaning against a wall. If the bottom of the ladder is pulled along the level pavement directly away from the wall at 4 feet per second, how fast is the top of the ladder moving down the wall when the foot of the ladder is 6 feet from the wall?

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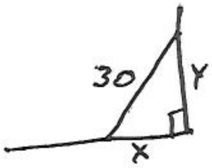

$\frac{dx}{dt} = 4$

Find $\left.\frac{dY}{dt}\right|_{x=6}$

$x^2 + Y^2 = 900$

$2x\frac{dx}{dt} + 2Y\frac{dY}{dt} = 0$

$\frac{dY}{dt} = \frac{-x}{Y}\frac{dy}{dt}$

when $x = 6$ $Y = \sqrt{900 - 36}$

$\frac{dY}{dt} = \frac{-6}{\sqrt{864}}(4)$ ft/sec.

.8 ft/sec

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10. Find $\int (\frac{2}{x^4} + \sqrt{2-3x}\,)\,dx$. Check your answer.

$$\boxed{-\frac{2}{3x^3} + \frac{2}{3}\left(\frac{-1}{3}\right)(2-3x)^{\frac{3}{2}} + C}$$

10

11. Find $\int x^2 \sin(8x^3+18)\,dx$. Check your answer.

$$\boxed{\frac{-1}{24}\cos(8x^3+18) + C}$$

10

12. Find $\int_0^1 \frac{x^2}{\sqrt{4x^3+18}}\,dx$. $= \int_{18}^{22} \frac{1}{12} u^{-\frac{1}{2}}\,du = \frac{1}{6} u^{\frac{1}{2}}\Big]_{18}^{22}$

$\uparrow$

$u = 4x^3+18$

$du = 12x^2 dx$

$$= \boxed{\frac{1}{6}(\sqrt{22}-\sqrt{18})}$$

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13. Let

$$f(x) = 3x - x^3.$$

Find where $f(x)$ is increasing, decreasing, concave up, and concave down. Find the local extreme points and the points of inflection of $y = f(x)$. Find the vertical and horizontal asymptotes of $y = f(x)$. GRAPH $y = f(x)$.

$f' = 3 - 3x^2 = -3(x^2-1)$

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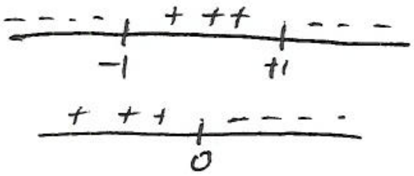
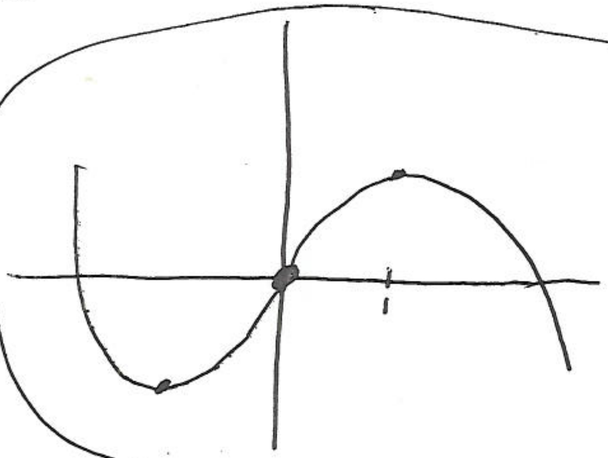
$f'' = -6x$





l. Max (1,2)

l. Min (-1,-2)

inc. $-1<x<1$

dec $x<-1$ also $1<x$

c.u $x<0$

c.d. $0<x$

No asy

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14. Find the area of the region which is bounded by $y = x$, $y + x^2 = 0$ and $x = 2$.

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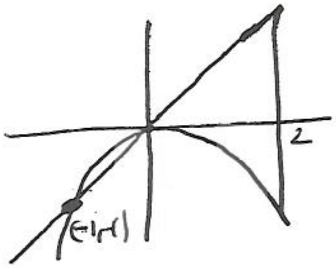

$\int_{-1}^{0} -x^2-x\,dx + \int_{0}^{2} x+x^2\,dx$

$= -\frac{x^3}{3}-\frac{x^2}{2}\Big]_{-1}^{0} + \frac{x^2}{2}+\frac{x^3}{3}\Big]_{0}^{2}$

$= -\left(\frac{1}{3}-\frac{1}{2}\right) + 2 + \frac{8}{3}$

$= -\frac{2}{6}+\frac{3}{6}+\frac{12}{6}+\frac{16}{6} = \boxed{\frac{29}{6}}$

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$\int_{0}^{2} x+x^2\,dx$

$2+\frac{8}{3}$

$= \frac{14}{3}$

15. Let R be the region in the first quadrant which is bounded by $y = x^2$, $x = 2$, and the $x-$ axis. Find the volume of the solid which is obtained by revolving R about the $x-$ axis.

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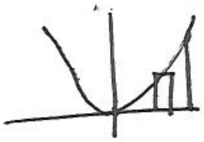

$\int_{0}^{2} \pi x^4\,dx = \frac{\pi x^5}{5}\Big]_{0}^{2} = \boxed{\frac{32\pi}{5}}$

$2\pi\left(16-\frac{2}{5}(4)^{5/2}\right)$

is ok also

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16. Find the length of $y = \frac{2}{3}(x^2+1)^{3/2}$ from $x = 1$ to $x = 4$.

$$\int_1^4 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_1^4 \sqrt{1+\left((x^2+1)^{\frac{1}{2}}\cdot 2x\right)^2}\,dx = \int_1^4 \sqrt{1+(x^2+1)4x^2}\,dx$$

$$= \int_1^4 \sqrt{4x^4+4x^2+1}\,dx = \int_1^4 \sqrt{(2x^2+1)^2}\,dx = \int_1^4 2x^2+1\,dx = \frac{2x^3}{3}+x\Big]_1^4$$

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~~$= 16+4-2 = 18$~~

$$\boxed{\frac{128}{3}+4-\frac{2}{3}-1}$$

17. Find the area of the surface obtained by revolving $y = \sqrt{25-x^2}$, from $x = -2$ to $x = 3$, about the $x-$axis.

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$$2\pi\int_{-2}^{3} \sqrt{25-x^2}\sqrt{1+\left(\frac{-2x}{2\sqrt{25-x^2}}\right)^2}\,dx$$

$$= 2\pi\int_{-2}^{3} \cancel{\sqrt{25-x^2}}\,\frac{\sqrt{4(25-x^2)+4x^2}}{\sqrt{4}\,\cancel{\sqrt{25-x^2}}}\,dx$$

$$= \frac{2\pi}{2}\int_{-2}^{3} 10\,dx$$

$$= \frac{2\pi}{2}\,10x\Big]_{-2}^{3}$$

$$= \cancel{\frac{2\pi}{2}\,50}$$

$$= \cancel{\boxed{100\pi}}$$

50π

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18. Let

$$f(x) = 16x^{-\frac{1}{3}} + x^{\frac{5}{3}}.$$

Find where $f(x)$ is increasing, decreasing, concave up, and concave down. Find the local extreme points and the points of inflection of $y = f(x)$. Find the vertical and horizontal asymptotes of $y = f(x)$. GRAPH $y = f(x)$.

$f' = \frac{-16}{3} x^{-\frac{4}{3}} + \frac{5}{3} x^{\frac{2}{3}} = \frac{1}{3} x^{-\frac{4}{3}} \left(-16 + 5x^2\right)$

$+ \quad | \quad - \quad | \quad +$ at $-\frac{4}{\sqrt{5}}$, $\frac{4}{\sqrt{5}}$

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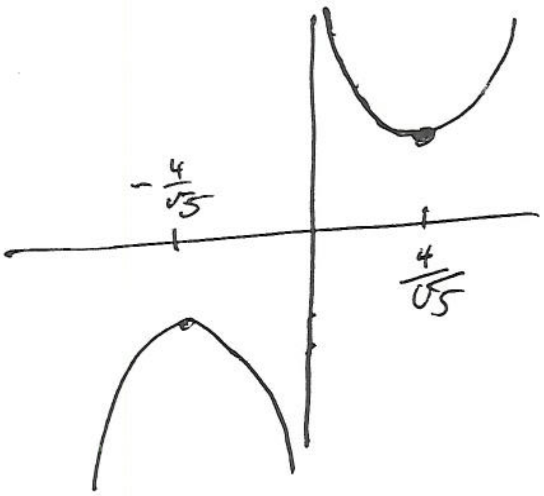
$f'' = \frac{64}{9} x^{-\frac{7}{3}} + \frac{10}{9} x^{-\frac{1}{3}} = \frac{1}{9} x^{-\frac{7}{3}} \left(64 + 10x^2\right)$

$- - - \quad | \quad + + +$ at 0



v. asy of $x=0$

l. min $\left(\frac{4}{\sqrt{5}} \; f\left(\frac{4}{\sqrt{5}}\right)\right)$

l. max $\left(-\frac{4}{\sqrt{5}}, \; f\left(-\frac{4}{\sqrt{5}}\right)\right)$

c.u. $0 < x$

c.d $x < 0$

dec $-\frac{4}{\sqrt{5}} < x < \frac{4}{\sqrt{5}}$

inc. $x < \frac{-4}{\sqrt{5}}$; $\frac{4}{\sqrt{5}} < x$

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19. An open box with a capacity of 72,000 cubic inches is needed. If the box must be twice as long as it is wide, what dimensions would require the least amount of material?

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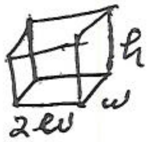

$72{,}000 = 2w^2h \quad \leftarrow \quad \frac{36{,}000}{w^2} = h$

$A = 2w^2 + 2wh + 4wh$

$A = 2w^2 + 6wh$

$A = 2w^2 + \frac{6 \cdot 36{,}000}{w}$

$A' = 4w - \frac{6 \cdot 36{,}000}{w^2} = \frac{4w^3 - 6 \cdot 6 \cdot 6 \cdot 10^3}{w^3}$

$A' = 0$ whe $w = \frac{60}{\sqrt[3]{4}} = 30\sqrt[3]{2} = \sqrt[3]{54000}$

$l = \frac{120}{\sqrt[3]{4}} = 60\sqrt[3]{2}$

$h = \frac{36{,}000}{\frac{3600}{\sqrt[3]{16}}} = 20\sqrt[3]{2}$