Math 141, Final Exam, Fall 2005, Solution

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.; although, by using enough paper, you can do the problems in any order that suits you.

There are 23 problems. Problems 1 through 7 are worth 8 points each. Each of the other problems is worth 9 points. The exam is worth 200 points. SHOW your work. Make your work be coherent and clear. Write in complete sentences whenever this is possible. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible. **No Calculators.**

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**. Otherwise, get your grade from VIP.

You might find the following formulas to be useful:

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \quad \text{and} \quad \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}.$$

I will post the solutions on my website a few hours after the exam is finished.

1. **Let** $y = 2^x$ **. Find** $\frac{dy}{dx}$ **.**

The derivative is $\boxed{\frac{dy}{dx} = (\ln 2)2^x.}$

2. **Let** $y = \cos(\cos x)$ **. Find** $\frac{dy}{dx}$ **.**

The derivative is $\boxed{\frac{dy}{dx} = \sin x \sin(\cos x)}$.

3. **Let** $y = x^2(\arcsin x)^3$ **. Find** $\frac{dy}{dx}$ **.**

The derivative is $\boxed{\frac{dy}{dx} = \frac{3x^2(\arcsin x)^2}{\sqrt{1-x^2}} + 2x(\arcsin x)^3}$.

4. **Let** $y = \sin x \left(\int_0^x \sin(t^2)dt\right)$ **. Find** $\frac{dy}{dx}$ **.**

The derivative is $\boxed{\frac{dy}{dx} = \sin x \sin(x^2) + \cos x \left(\int_0^x \sin(t^2)dt\right)}$.

5. **Find** $\int_1^2 x^2 dx$ **.**

The integral is

$$\left.\frac{x^3}{3}\right|_1^2 = \frac{8}{3} - \frac{1}{3} = \boxed{\frac{7}{3}}.$$

6. **Find** $\int_{\frac{\pi}{12}}^{\frac{\pi}{9}} \sec^2 3\theta d\theta$ **.**

The integral is

$$\left.\frac{\tan 3\theta}{3}\right|_{\frac{\pi}{12}}^{\frac{\pi}{9}} = \frac{\tan\frac{\pi}{3}}{3} - \frac{\tan\frac{\pi}{4}}{3} = \boxed{\tfrac{1}{3}(\sqrt{3}-1)}.$$

7. **Find** $\int_{-1}^{1} \frac{x^2 dx}{\sqrt{x^3+9}}$ **.**

The integral is

$$\left.\frac{2\sqrt{x^3+9}}{3}\right|_{-1}^{1} = \boxed{\frac{2\sqrt{10}}{3} - \frac{2\sqrt{8}}{3}.}$$

8. **Find** $\int_0^{\frac{\sqrt{\pi}}{2}} 5x\cos(x^2) dx$ **.**

The integral is

$$\left.\frac{5}{2}\sin(x^2)\right|_0^{\frac{\sqrt{\pi}}{2}} = \frac{5}{2}\left(\sin(\frac{\pi}{4}) - \sin 0\right) = \boxed{\frac{5\sqrt{2}}{4}.}$$

9. **Find** $\lim_{n\to\infty} \frac{1}{n^3}\sum_{k=1}^{n} k^2$ **.**

The limit is

$$\lim_{n\to\infty} \frac{1}{n^3}\frac{n(n+1)(2n+1)}{6} = \lim_{n\to\infty}\frac{(1+\frac{1}{n})(2+\frac{1}{n})}{6} = \boxed{\frac{1}{3}}.$$

10. **Find** $\lim\limits_{x\to\infty} x^2 - \sqrt{x^4+6x^2}$ **.**

The limit is

$$\lim_{x\to\infty} \frac{x^4-(x^4+6x^2)}{x^2+\sqrt{x^4+6x^2}} = \lim_{x\to\infty} \frac{-6x^2}{x^2+\sqrt{x^4+6x^2}} = \lim_{x\to\infty} \frac{-6}{1+\sqrt{1+\frac{6}{x^2}}} = \boxed{-3}.$$

11. **Find** $\lim\limits_{x\to 0^+} x^{\frac{5}{1+\ln x}}$ **.**

Let $y = x^{\frac{5}{1+\ln x}}$. We see that

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{5}{1+\ln x}\ln x = \lim_{x\to 0^+} \frac{5}{\frac{1}{\ln x}+1} = 5.$$

We conclude that $\lim\limits_{x\to 0^+} y = \lim\limits_{x\to 0^+} e^{\ln y} = \boxed{e^5}.$

12. **Use the DEFINITION of the derivative to find** $f'(x)$ **for** $f(x) = \frac{1}{\sqrt{2x-3}}$ **.**

We know that

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{\sqrt{2(x+h)-3}} - \frac{1}{\sqrt{2x-3}}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{2x-3}-\sqrt{2(x+h)-3}}{h\sqrt{2(x+h)-3}\sqrt{2x-3}}$$

$$= \lim_{h\to 0} \frac{2x-3-(2(x+h)-3)}{h\sqrt{2(x+h)-3}\sqrt{2x-3}(\sqrt{2x-3}+\sqrt{2(x+h)-3})}$$

$$= \lim_{h\to 0} \frac{-2}{\sqrt{2(x+h)-3}\sqrt{2x-3}(\sqrt{2x-3}+\sqrt{2(x+h)-3})}$$

$$= \frac{-2}{\sqrt{2x-3}\sqrt{2x-3}(\sqrt{2x-3}+\sqrt{2x-3})} = \frac{-1}{\sqrt{2x-3}\sqrt{2x-3}\sqrt{2x-3}}$$

$$= \boxed{-(2x-3)^{-\frac{3}{2}}.}$$

13. **Parameterize the diamond with vertices $(1,0)$, $(0,1)$, $(-1,0)$, and $(0,-1)$.**

We start at $t=0$ at the point $(1,0)$ and we move in a counter clock wise direction. First we walk on the line $y=1-x$ from $t=0$ to $t=1$. Then we walk on $y=x+1$ from $t=1$ to $t=2$. Then we walk on $y=-x-1$ from $t=2$ to $t=3$. Finally, we walk on $y=x-1$ from $t=3$ to $t=4$. So our parametrization is:

$$\boxed{x=\begin{cases}1-t & \text{for } 0\le t<1\\ 1-t & \text{for } 1\le t<2\\ t-3 & \text{for } 2\le t<3\\ t-3 & \text{for } 3\le t\le 4\end{cases}\quad \text{and} \quad y=\begin{cases}t & \text{for } 0\le t<1\\ 2-t & \text{for } 1\le t<2\\ 2-t & \text{for } 2\le t<3\\ t-4 & \text{for } 3\le t\le 4\end{cases}}$$

A picture appears of a different page.

14. **The position of an object at time t is given by**

$$\begin{cases} x=4\sin t\\ y=3\cos t.\end{cases}$$

(a) Eliminate the parameter to find a Cartesian equation for the path of the object.
(b) Graph the path of the object.
(c) On your graph, mark the position of the object at a few particular values for time.

We know that $\sin^2 t+\cos^2 t=1$; so, the answer to (a) is $\boxed{\dfrac{x^2}{16}+\dfrac{y^2}{9}=1.}$ The answer to (b) is the ellipse with vertices $(0,3)$ (at time $t=0$), $(4,0)$ (at time $t=\frac{\pi}{2}$), $(0,-3)$ (at $t=\pi$), and $(-4,0)$ (at $t=\frac{3\pi}{2}$). So the object travels around the ellipse in a clockwise manner. The picture appears on a different page.

15. **Let $f(x)=4x^{1/3}-x^{4/3}$. Where is $f(x)$ increasing, decreasing, concave up, and concave down? What are the local extreme points and points of inflection of $y=f(x)$. Find all vertical and horizontal asymptotes. Graph $y=f(x)$.**

We see that

$$f'(x)=\tfrac{4}{3}x^{-2/3}-\tfrac{4}{3}x^{1/3}=\tfrac{4}{3}x^{-2/3}(1-x).$$

So $f'(x)=0$ when $x=1$ and $f'(x)$ does not exist when $x=0$. We see that $f'(x)\le 0$ for $1<x$, and $f'(x)\ge 0$ for $x<0$, also for $0<x<1$. We conclude that:

$f(x)$ is increasing for $x<1$ and $f(x)$ is decreasing for $1<x$, $(1,3)$ is the only local maximum point on the graph, and there are no local minimum points on the graph.

We also see that

$$f''(x) = \tfrac{4}{3}(-\tfrac{2}{3}x^{-5/3} - \tfrac{1}{3}x^{-2/3}) = -\tfrac{4}{9}x^{-5/3}(2+x).$$

Thus, $f''(x) = 0$ when $x = -2$; $f''(x)$ does not exist for $x = 0$. We see that $f''(x) \geq 0$ for $-2 < x < 0$ and that $f''(x) < 0$ for $x < -2$, also for $0 < x$. We conclude that

$$\boxed{\begin{array}{l} f(x) \text{ is concave up for } -2 < x < 0, \\ f(x) \text{ is concave down for } x < -2, \text{ also for } 0 < x, \\ (-2, f(-2)) \text{ and } (0,0) \text{ are the points of inflection.} \end{array}}$$

There are no asymptotes. As x goes to $+\infty$ or $-\infty$, the graph goes to $-\infty$; so, no vertical asymptotes. There are no numbers c with $\lim\limits_{x\to c} f(x)$ equal to $\pm\infty$; so, no horizontal asymptotes. The graph appears on a different piece of paper.

16. **Each edge of a cube is increasing at the rate of 4 inches per second. How fast is the surface area of the cube increasing when an edge is 12 inches long?**

Let $\ell(t)$ the the length of each edge of the cube at time t and $S(t)$ be the surface area of the cube at time t. We are told that $\frac{d\ell}{dt} = 4$ in/sec. We want $\frac{dS}{dt}\Big|_{\ell=12}$. We know that $S = 6\ell^2$. So $\frac{dS}{d\ell} = 12\ell\frac{d\ell}{dt}$. We conclude that

$$\frac{dS}{dt}\bigg|_{\ell=12} = \boxed{(12)(12)4\text{in}^2/\text{sec}}$$

17. **Consider the right circular cylinder of greatest volume that can be inscribed in a right circular cone. What is the ratio of the volume of the cylinder divided by the volume of the cone ?**

A picture appears on a different page. Let r_0 be the radius of the base of the cone and h_0 be the height of the cone. (Notice that r_0 and h_0 are constants.) Let r be the radius of the base of cylinder and h be the height of the cylinder. (Notice that we get to vary r and h.) Our job is to maximize V, which is the volume of the cylinder. We know that $V = \pi r^2 h$. Let α be the ratio r/r_0. We express everything in terms of the one variable α. Similar triangles (see the other page) shows us that $\frac{r}{r_0} = \frac{h_0 - h}{h_0}$; so, $\alpha = 1 - \frac{h}{h_0}$; that is $\frac{h}{h_0} = 1 - \alpha$, or $h = (1-\alpha)h_0$. We now know that $V = \pi r_0^2 h_0 \alpha^2(1-\alpha)$. The number $\pi r_0^2 h_0$ is a constant. The value of V is maximized when $W(\alpha) = \alpha^2(1-\alpha)$ is maximized for $0 < \alpha < 1$. We write $W = \alpha^2 - \alpha^3$. We see that $W' = 2\alpha - 3\alpha^2 = \alpha(2 - 3\alpha)$. So $W'(\alpha)$ is equal to zero when $\alpha = 0$ or $\alpha = \frac{2}{3}$. The graph of $W(\alpha)$ is zero at the endpoints, so W is maximized at the unique point in the domain of α where $W'(\alpha) = 0$; namely, at $\alpha = \frac{2}{3}$. For this choice of α, the volume of the cylinder is

$$\pi r_0^2 h_0 \frac{4}{9}\frac{1}{3}.$$

The volume of the cone is equal to $\frac{1}{3}\pi r_0^2 h_0$. Thus, the ratio of the volume of the cylinder divided by the volume of the cone of maximum volume is

$$\frac{\pi r_0^2 h_0 \frac{4}{9}\frac{1}{3}}{\frac{1}{3}\pi r_0^2 h_0} = \boxed{\frac{4}{9}}.$$

18. **State the Mean Value Theorem.**

If f is a continuous function on the closed interval $[a, b]$, with f differentiable on the open interval (a, b), then there exists a number $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

19. **Consider the region bounded by $y = x^2$, $x = 1$, $x = 2$, and the x-axis. Partition the base into 50 subintervals of equal size. Over each subinterval, imagine a rectangle which approximates, but OVER estimates, the area under the curve. How much area is inside your 50 rectangles? (You must answer the question I asked, not some other question. I expect an exact answer in closed form: no dots and no summation signs.)**

We have partitioned the closed interval $[1, 2]$ into the 50 subintervals: $[1 + \frac{k-1}{50}, 1 + \frac{k}{50}]$, for $1 \le k \le 50$. The function $f(x) = x^2$ is increasing in the region under consideration, so the maximum value of $f(x)$ over the k^{th}-subinterval $[1+\frac{k-1}{50}, 1+\frac{k}{50}]$ occurs at the right end point $1+\frac{k}{50}$. The area of the k^{th}-rectangle is $\frac{1}{50}(1 + \frac{k}{50})^2$. The area inside all 50 rectangles is

$$\sum_{k=1}^{50} \frac{1}{50}\left(1 + \frac{k}{50}\right)^2 = \frac{1}{50}\sum_{k=1}^{50}\left(1 + 2\frac{k}{50} + \frac{k^2}{50^2}\right)$$

$$= \frac{1}{50}\left(\sum_{k=1}^{50} 1 + \frac{2}{50}\sum_{k=1}^{50} k + \frac{1}{50^2}\sum_{k=1}^{50} k^2\right)$$

$$= \boxed{\frac{1}{50}\left(50 + \frac{2}{50}\frac{50(51)}{2} + \frac{1}{50^2}\frac{50(51)(101)}{6}\right).}$$

20. **The position of an object above the surface of the earth is given by**

$$s(t) = -16t^2 + 64t + 100,$$

where s **is measured in feet and** t **is measured in seconds. How high does the object get?**

The object reaches its maximum height when $s'(t) = 0$. We see that $s'(t) = -32t + 64$. Thus, $s'(t) = 0$ when $t = 2$ and the height at this moment is $s(2) = -64 + 2(64) + 100 = \boxed{164 \text{ ft}}$

21. **State BOTH parts of the Fundamental Theorem of Calculus.**

Let f be a continuous function defined on the closed interval $[a, b]$.
(a) If $A(x)$ is the function $A(x) = \int_a^x f(t)dt$, for all $x \in [a, b]$, then $A'(x) = f(x)$ for all $x \in [a, b]$.
(b) If $F(x)$ is any antiderivative of $f(x)$, then $\int_a^b f(x)dx = F(b) - F(a)$.

22. **Let** a **and** b **be real numbers with** $\frac{-\pi}{2} < a < b < \frac{\pi}{2}$**. Prove that** $\tan b - \tan a \ge b - a$**.**

We apply the Mean Value Theorem. We notice that the function $\tan x$ is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b); so, there exists a number c in (a, b) with

$$\frac{\tan b - \tan a}{b - a} = \sec^2 c.$$

Of course, the value of $\sec^2 c$ is always at least one. Thus,

$$\frac{\tan b - \tan a}{b - a} \ge 1.$$

Multiply both sides of the inequality by the positive number $b - a$ to get $\tan b - \tan a \ge b - a$.

23. **Find the equations of the lines through the origin that are tangent to** $2x^2 - 4x + y^2 + 1 = 0$**.**

Let $P = (a, b)$ be a point on the curve. Take $\frac{d}{dx}$ of both sides of the equation of the curve to see that $4x - 4 + 2y\frac{dy}{dx} = 0$. So, $\frac{dy}{dx} = \frac{2-2x}{y}$, and the slope of the line tangent to the curve at P is $\frac{dy}{dx}|_P = \frac{2-2a}{b}$. The equation of the line tangent to the curve at P is $y - b = \frac{2-2a}{b}(x - a)$. We hope to find all points P which are on the curve and also have $(0,0)$ sit on the line tangent to the curve at P. We must solve

$$2a^2-4a+b^2+1=0 \quad \text{and} \quad -b=\frac{2-2a}{b}(-a)$$

simultaneously. We see that

$$b^2=4a-2a^2-1 \quad \text{and} \quad b^2=2a-2a^2.$$

So, $4a-2a^2-1=2a-2a^2$; that is, $2a=1$, or $a=\frac{1}{2}$. Once we know a, then we know that $b^2=1-2(\frac{1}{4})$; so, $b=\pm\frac{1}{\sqrt{2}}$. The line tangent to the curve at $(\frac{1}{2},\frac{1}{\sqrt{2}})$ is $y-\frac{1}{\sqrt{2}}=\sqrt{2}(x-\frac{1}{2})$ or $\boxed{y=\sqrt{2}x}$. The line tangent to the curve at $(\frac{1}{2},\frac{-1}{\sqrt{2}})$ is $y+\frac{1}{\sqrt{2}}=-\sqrt{2}(x-\frac{1}{2})$ or $\boxed{y=-\sqrt{2}x}$.