Math 141, Exam 4, Fall 2005 Solutions

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.; although, by using enough paper, you can do the problems in any order that suits you.

There are 12 problems. Problems 1 through 4 are worth 9 points each. Problems 5 through 12 are worth 8 points each. The exam is worth 100 points. SHOW your work. Make your work be coherent and clear. Write in complete sentences whenever this is possible. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible. **No Calculators.**

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

I will post the solutions on my website a few hours after the exam is finished.

1. **Find** $\int \frac{dx}{e^x}$ **. Check your answer.**

The problem is equal to $\int e^{-x}dx = \boxed{-e^{-x}+C}$.

2. **Find** $\int \sec 4x \tan 4x dx$ **. Check your answer.**

Let $u = 4x$; so, $du = 4dx$. The problem is equal to

$$\tfrac{1}{4}\int \sec u \tan u \, du = \tfrac{1}{4}\sec u = \boxed{\frac{1}{4}\sec 4x + C.}$$

Check. The derivative of the proposed answer is

$$\tfrac{1}{4}4\sec 4x \tan 4x. \checkmark$$

3. **Find** $\int \frac{\sec^2 x dx}{\sqrt{1-\tan^2 x}}$. **Check your answer.**

Let $u = \tan x$; so, $du = \sec^2 x dx$. The problem is equal to

$$\int \tfrac{du}{\sqrt{1-u^2}} = \arcsin u + C = \boxed{\arcsin(\tan x) + C}.$$

Check. The derivative of the proposed answer is

$$\tfrac{\sec^2 x}{\sqrt{1-\tan^2 x}}. \checkmark$$

4. **Find** $\lim\limits_{\Delta x\to 0} \frac{\ln(e^2+\Delta x)-2}{\Delta x}$ **.**

The top and the bottom both go to zero. We apply L'Hôpital's rule to see that the given limit is equal to

$$\lim_{\Delta x\to 0} \frac{\frac{1}{e^2+\Delta x}}{1} = \boxed{\frac{1}{e^2}}.$$

5. **Find** $\lim\limits_{x\to 0^+} x^{\frac{\ln 2}{1+\ln x}}$.

The base goes to 0. The exponent goes to 0. This is an indeterminate form. We must be clever. Let $y = x^{\frac{\ln 2}{1+\ln x}}$. We see that

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{\ln 2}{1+\ln x}\ln x = \lim_{x\to 0^+} \frac{\ln 2}{\frac{1}{\ln x}+1} = \ln 2.$$

It follows that the answer is

$$\lim_{x\to 0^+} y = \lim_{x\to 0^+} e^{\ln y} = e^{\ln 2} = \boxed{2}.$$

6. **Find** $\lim\limits_{x\to +\infty} \frac{x^3}{e^{-x}}$ **.**

This limit is $\lim\limits_{x\to +\infty} x^3e^x$. Both factors head to $+\infty$. There is no conflict. The answer is $\boxed{+\infty}$.

7. **Find** $\frac{dy}{dx}$ **for** $\sin(x^2y^2) = x$ **.**

Take $\frac{d}{dx}$ of both sides to get

$$(x^2 2y\frac{dy}{dx} + 2xy^2)\cos(x^2y^2) = 1.$$

Thus,

$$\boxed{\frac{dy}{dx} = \frac{1-2xy^2\cos(x^2y^2)}{2x^2y\cos(x^2y^2)}.}$$

8. **Find** $\frac{dy}{dx}$ **for** $y = \ln(\sin^2 x)$ **.**

We rewrite y as $y = 2\ln(\sin x)$. We compute $\boxed{\frac{dy}{dx} = \frac{2\cos x}{\sin x}.}$

9. **Find $\frac{dy}{dx}$ for $y = x^{\sin x}$.**

Take ln of both sides to get

$$\ln y = \sin x \ln x.$$

Take $\frac{d}{dx}$ of both sides to get

$$\frac{1}{y}\frac{dy}{dx} = \frac{\sin x}{x} + \cos x \ln x.$$

Solve for $\frac{dy}{dx}$:

$$\boxed{\frac{dy}{dx} = y(\frac{\sin x}{x} + \cos x \ln x).}$$

10. **Find the coordinates of the point P on the curve**

$$y = \frac{1}{x^2} \quad \textbf{for } x > 0$$

where the segment of the tangent line at P that is cut off by the coordinate axes has its shortest length.

Let $P = (a, \frac{1}{a^2})$ be a point on the curve. The derivative of the equation for the curve is $\frac{dy}{dx} = -2\frac{1}{x^3}$. So, the slope of the line tangent to the curve at P is $\frac{-2}{a^3}$ and the equation of the line tangent to the curve at P is $y - \frac{1}{a^2} = \frac{-2}{a^3}(x - a)$. The equation of the tangent line may be rewritten as

$$y = \frac{-2x}{a^3} + \frac{3}{a^2}.$$

The tangent line hits the x-axis, when $y = 0$; so $0 = -2x + 3a$. This is the point $Q = (\frac{3a}{2}, 0)$. The tangent line hits the y-axis when $x = 0$, so $y = \frac{3}{a^2}$. This is the point $R = (0, \frac{3}{a^2})$. Our job is to pick a, with $a > 0$ so that the distance from Q to R is minimized. The distance from Q to R is

$$D = \sqrt{\left(\frac{3a}{2}\right)^2 + \left(\frac{3}{a^2}\right)^2}.$$

We notice that D is minimized when the expression U, which sits under the radical, is minimized. Our job is to minimize

$$U(a) = \left(\frac{3a}{2}\right)^2 + \left(\frac{3}{a^2}\right)^2, \quad \text{for } a > 0.$$

We write $U(a) = \frac{9}{4}a^2 + 9a^{-4}$. We see that U goes to infinity as a goes to zero and U goes to infinity as a goes to infinity. The minimum value of U will be attained at the point where $U'(a) = 0$. We calculate:

$$U'(a) = \tfrac{9}{2}a - 36a^{-5} = \tfrac{9}{2}a^{-5}(a^6 - 8) = \tfrac{9}{2}a^{-5}(a^2 - 2)(a^4 + 2a^2 + 4).$$

The factor a^4+2a^2+4 is always greater than zero (and hence never equal to zero). The only positive real number a with $U'(a) = 0$ is $a = \sqrt{2}$.

$\boxed{\text{The point } P \text{ on the curve for which the segment of the tangent line at } P \text{ that is cut off by the coordinate axes has shortest length is } P = (\sqrt{2}, \tfrac{1}{2}).}$

I put a picture on the last page.

11. **Let $f(x) = x^2 \ln x$. Where is $f(x)$ increasing, decreasing, concave up, and concave down? What are the local maximum points, local minimum points, and points of inflection of $y = f(x)$. Find all vertical and horizontal asymptotes. What is the domain of $f(x)$? Graph $y = f(x)$.**

$\boxed{\text{The domain of } f(x) \text{ is all positive real numbers } x}$ because the domain of $\ln x$ is all positive real numbers x . We see that

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^2 \ln x = \lim_{x\to 0^+} \frac{\ln x}{x^{-2}}.$$

The top and the bottom both go to infinity so L'Hôpital's rule gives that

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-2x^{-3}} = \lim_{x\to 0^+} \frac{x^2}{-2} = 0.$$

Thus, the graph $y = f(x)$ never goes to infinity as x approaches a number. We conclude that

$\boxed{\text{There are no vertical asymptotes.}}$

There is no difficulty taking the limit as x goes to ∞ of $f(x)$. Both factors go to ∞ , so the limit is $+\infty$ and we conclude that

$\boxed{\text{There are no horizontal asymptotes.}}$

Take the derivative to see

$$f'(x) = x^2\frac{1}{x} + 2x\ln x = x(1 + 2\ln x).$$

We see that $f'(x)$ is zero (in the domain of f) only at $x = e^{-\frac{1}{2}}$. Thus,

$$\boxed{\begin{array}{l} f(x) \text{ is decreasing for } 0 < x < e^{-\frac{1}{2}}, \\ f(x) \text{ is increasing for } e^{-\frac{1}{2}} < x, \\ (e^{-\frac{1}{2}}, f(e^{-\frac{1}{2}})) \text{ is the only local minimum point, and} \\ \text{there are no local maximum points.} \end{array}}$$

Take the derivative to see that

$$f''(x) = x\frac{2}{x} + (1 + 2\ln x) = 3 + 2\ln x.$$

Thus, $f'(x) = 0$ at $x = e^{\frac{-3}{2}}$. Thus,

$$\boxed{\begin{array}{l} f(x) \text{ is concave down } 0 < x < e^{-\frac{3}{2}}, \\ f(x) \text{ is concave up for } e^{-\frac{3}{2}} < x, \text{ and} \\ (e^{-\frac{1}{2}}, f(e^{-\frac{3}{2}})) \text{ is the only point of inflection.} \end{array}}$$

I put the picture on the last page.

12. **A boat is pulled into a dock by means of a rope attached to a pulley on the dock. (See the picture on the last page.) The rope is attached to the bow of the boat at a point 10 feet below the pulley. If the rope is pulled through the pulley at a rate of 20 feet/minute, at what rate will the boat be approaching the dock when 125 feet of rope are out?**

Let x be the distance from the boat to the dock. Let z be the amount of rope out. We have a right triangle with vertical side 10, horizontal side x, and hypotenuse z, see the picture on the last page. So, $x^2 + 100 = z^2$. We are told that $\frac{dz}{dt} = -20$. We want $\frac{dx}{dt}$, when $z = 125$. $2x\frac{dx}{dt} = 2z\frac{dz}{dt}$. When $z = 125$, then $x = \sqrt{(125)^2 - 100}$ and at that moment,

$$\boxed{\left.\frac{dx}{dt}\right|_{z=125 \text{ feet}} = \frac{125}{\sqrt{(125)^2 - 100}}(-20)\frac{\text{feet}}{\text{minute}}.}$$