Math 141, Exam 3, Fall 2005 Solutions

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.; although, by using enough paper, you can do the problems in any order that suits you.

There are 14 problems. Problem 1 is worth 9 points. Each other problem is worth 7 points. The exam is worth 100 points. SHOW your work. Make your work be coherent and clear. Write in complete sentences whenever this is possible. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible. **No Calculators.**

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

I will post the solutions on my website a few hours after the exam is finished.

1. **Let** $f(x) = \begin{cases} x^2 & \textbf{for } x < 1, \\ 2x-1 & \textbf{for } 1 \le x \le 2 \\ 5-x & \textbf{for } 2 < x < 3 \\ x & \textbf{for } 3 \le x. \end{cases}$

   (a) **Graph** $y = f(x)$ **.**

The graph appears on a separate piece of paper. Be sure to notice that the graph is continuous and differentiable at $x = 1$. The line tangent to $y = x^2$ at $x = 1$ is $y = 2x - 1$. The graph is continuous, but not differentiable at $x = 2$, there is a sharp turn. The graph is neither continous nor differentiable at $x = 3$

   (b) **Find**

$\lim_{x\to 0^+} f(x) = 0$	$\lim_{x\to 0^-} f(x) = 0$	$\lim_{x\to 0} f(x) = 0$	$f(0) = 0$
$\lim_{x\to 1^+} f(x) = 1$	$\lim_{x\to 1^-} f(x) = 1$	$\lim_{x\to 1} f(x) = 1$	$f(1) = 1$
$\lim_{x\to 2^+} f(x) = 3$	$\lim_{x\to 2^-} f(x) = 3$	$\lim_{x\to 2} f(x) = 3$	$f(2) = 3$
$\lim_{x\to 3^+} f(x) = 3$	$\lim_{x\to 3^-} f(x) = 2$	$\lim_{x\to 3} f(x) = \textbf{DNE}$	$f(3) = 3$
$\lim_{x\to 4^+} f(x) = 4$	$\lim_{x\to 4^-} f(x) = 4$	$\lim_{x\to 4} f(x) = 4$	$f(4) = 4$

   (c) **Where is** $f(x)$ **continuous?**

The function $f(x)$ is continuous everywhere, except at $x = 3$.

   (d) **Where is** $f(x)$ **differentiable?**

The function $f(x)$ is differentiable everywhere, except at $x = 2$ and $x = 3$.

2. **Find the derivative of** $f(x) = 3x^2 + \sqrt{2x} + \frac{1}{4x} + e$ **.**

$$\boxed{f'(x) = 6x + \frac{2}{2\sqrt{2x}} - \frac{4}{(4x)^2}.}$$

3. **Parameterize the triangle whose vertices are $(1,0)$, $(0,1)$, and $(-1,0)$.**

I will start at at $t=0$ at $(1,0)$. I will walk along the triangle in a counter clockwise manner. I will reach $(0,1)$ at $t=1$, $(-1,0)$ at $t=2$, and I will return to $(1,0)$ at $t=3$. (One could take other trips around this triangle.)

$$\boxed{x(t)=\begin{cases}1-t & \text{if } 0\le t\le 1\\ 1-t & \text{if } 1\le t\le 2\\ 2(t-3)+1 & \text{if } 2\le t\le 3\end{cases} \text{ and } y(t)=\begin{cases}t & \text{if } 0\le t\le 1\\ 2-t & \text{if } 1\le t\le 2\\ 0 & \text{if } 2\le t\le 3\end{cases}}$$

For each leg of my journey I merely parameterized the line segment in question.

4. **Use the definition of the derivative to find the derivative of $f(x)=\sqrt{2x-3}$.**

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\sqrt{2(x+h)-3}-\sqrt{2x-3}}{h}$$

$$=\lim_{h\to 0}\frac{(\sqrt{2(x+h)-3}-\sqrt{2x-3})(\sqrt{2(x+h)-3}+\sqrt{2x-3})}{h(\sqrt{2(x+h)-3}+\sqrt{2x-3})}$$

$$=\lim_{h\to 0}\frac{(2(x+h)-3-(2x-3))}{h(\sqrt{2(x+h)-3}+\sqrt{2x-3})}=\lim_{h\to 0}\frac{2x+2h-3-2x+3}{h(\sqrt{2(x+h)-3}+\sqrt{2x-3})}$$

$$=\lim_{h\to 0}\frac{2h}{h(\sqrt{2(x+h)-3}+\sqrt{2x-3})}=\lim_{h\to 0}\frac{2}{(\sqrt{2(x+h)-3}+\sqrt{2x-3})}$$

$$=\frac{2}{(\sqrt{2x-3}+\sqrt{2x-3})}=\frac{2}{2\sqrt{2x-3}}=\boxed{\frac{1}{\sqrt{2x-3}}.}$$

5. **Find $\lim\limits_{n\to\infty}\left(\frac{n}{n-3}\right)^n$.**

Let $t=n-3$. Notice that as n goes to ∞, so does t. We see that

$$\lim_{n\to\infty}\left(\frac{n}{n-3}\right)^n=\lim_{t\to\infty}\left(\frac{t+3}{t}\right)^{t+3}=\lim_{t\to\infty}(1+3/t)^t\lim_{t\to\infty}(1+3/t)^3.$$

We know that the limit of a product is the product of the limits. We also know that $\lim\limits_{t\to\infty}\left(1+\frac{r}{t}\right)^t=e^r$. The answer to this problem is $\boxed{e^3}$.

6. **Find $\lim\limits_{x\to 0}\frac{1-\cos 3x}{x^2}$.**

Multiply top and bottom by $1+\cos 3x$ to see that our problem is equal to

$$\lim_{x\to 0}\frac{1-\cos^2 3x}{x^2(1+\cos 3x)}=\lim_{x\to 0}\frac{\sin^2 3x}{x^2(1+\cos 3x)}=\lim_{x\to 0}9\frac{\sin 3x}{3x}\frac{\sin 3x}{3x}\frac{1}{(1+\cos 3x)}.$$

We know that $\lim\limits_{t\to 0}\frac{\sin t}{t}=1$. Apply this fact twice with $3x$ playing the role of t to conclude that the limit is $\boxed{\frac{9}{2}}$.

7. **Find** $\lim_{x\to\infty} \sqrt{x^6+5x^3} - x^3$ **.**

Multiply top and bottom by $\sqrt{x^6+5x^3}+x^3$ to see that our problem is equal to

$$\lim_{x\to\infty} \frac{x^6+5x^3-x^6}{\sqrt{x^6+5x^3}+x^3}$$

$$= \lim_{x\to\infty} \frac{5x^3}{\sqrt{x^6+5x^3}+x^3}.$$

Divide top and bottom by x^3 . Recall that $x^3=\sqrt{x^6}$ for positive x . Our limit is

$$= \lim_{x\to\infty} \frac{5}{\sqrt{1+\frac{5}{x^3}}+1} = \boxed{\frac{5}{2}}.$$

8. **Find** $\lim_{x\to 1} \frac{x^6-1}{x-1}$ **.**

Divide. The limit is equal to

$$\lim_{x\to 1}(x^5+x^4+x^3+x^2+x+1) = \boxed{6}.$$

A cute way to do the problem is to notice that the limit computes $f'(1)$, where $f(x)=x^6$. Again, the answer is 6 .

9. **Find the equation of the line tangent to** $x^4+y^4=16$ **at** $(1, \sqrt[4]{15})$ **.**

Take $\frac{d}{dx}$ of both sides $4x^3+4y^3\frac{dy}{dx}=0$. So, $\frac{dy}{dx}=\frac{-x^3}{y^3}$. So $\frac{dy}{dx}\Big|_{(1,\sqrt[4]{15})} = \frac{-1}{(\sqrt[4]{15})^3}$ and the tangent line is

$$\boxed{y-\sqrt[4]{15} = \frac{-1}{(\sqrt[4]{15})^3}(x-1).}$$

10. **Find the derivative of** $f(x)=\sin(\ln(2x^2+3x))$ **.**

$$\boxed{f'(x) = \frac{(4x+3)\cos(\ln(2x^2+3x))}{2x^2+3x}.}$$

11. **Find the derivative of** $f(x)=e^{3x^2+2x}\tan x$ **.**

$$\boxed{f'(x) = e^{3x^2+2x}\sec^2 x + (6x+2)e^{3x^2+2x}\tan x.}$$

12. **Find the x-coordinates of all points on the graph of $y = 1 - x^2$ at which the tangent line passes through the point $(2, 0)$.**

Let $(a, 1-a^2)$ be a point on the parabola. We see that $y' = -2x$. So the slope of the line tangent to $y = 1 - x^2$ at the point $(a, 1-a^2)$ is $-2a$ and the tangent line is $y - (1-a^2) = -2a(x-a)$. We want to find all a so that the tangent line passes through $(2,0)$. So we want all a for which $(2,0)$ satisfies the equation of the tangent line:

$$0 - (1-a^2) = -2a(2-a)$$
$$-1 + a^2 = -4a + 2a^2$$
$$0 = a^2 - 4a + 1.$$

Use the quadratic formula to find a:

$$a = \frac{4 \pm \sqrt{16-4}}{2} = \frac{4 \pm 2\sqrt{3}}{2} = 2 \pm \sqrt{3}.$$

The points we are to find have x-cordinate $\boxed{2+\sqrt{3}}$ or $\boxed{2-\sqrt{3}}$.

13. **The height of an object above the ground at time t is $s(t) = -16t^2 + 32t + 48$, where s is measured in feet and t is measured in seconds. What is the velocity of the object when it strikes the ground?**

The object hits the ground when $s(t) = 0$; so $0 = -16(t^2 - 2t - 3) = -16(t-3)(t+1)$. So the object hits the ground when $t = 3$ or $t = -1$. (Of course, we are not interested in $t = -1$.) $v(t) = s'(t) = -32t + 32$. The velocity when the object hits the ground is $s'(3) = -32(3) + 32 = \boxed{-64 \text{ feet/second}}$

14. **A cube is growing at the constant rate of 1000 cubic inches per second. How fast is the surface area growing when each edge is 5 inches long?**

Let V be the volume of the cube at time t, A be the surface area of the cube at time t, and ℓ the the length of each edge at time t. We are told $\frac{dV}{dt} = 1000 \text{ in}^3/\text{sec}$. We want $\frac{dA}{dt}|_{\ell=5\text{in}}$. We know $V = \ell^3$ and $A = 6\ell^2$. There are 6 faces: top, bottom, left, right, back and front. So, $\frac{dV}{dt} = 3\ell^2\frac{d\ell}{dt}$ and $\frac{dA}{dt} = 12\ell\frac{d\ell}{dt}$. It follows that

$$\frac{dA}{dt} = 12\ell\frac{dV}{dt}\frac{1}{3\ell^2} = \frac{dV}{dt}\frac{4}{\ell}$$

and

$$\frac{dA}{dt}|_{\ell=5\text{in}} = 1000 \text{ in}^3/\text{sec}\frac{4}{5 \text{ in}} = \boxed{800 \text{ in}^2/\text{sec.}}$$