Math 141, Exam 2, Fall 2005

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.; although, by using enough paper, you can do the problems in any order that suits you.

There are 10 problems; each problem is worth 10 points. SHOW your work. Make your work be coherent and clear. Write in complete sentences whenever this is possible. *CIRCLE* your answer. **CHECK** your answer whenever possible. **No Calculators.**

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

I will post the solutions on my website shortly a few hours after the exam is finished.

1. **Find** $\lim\limits_{x\to 0^-} (1+3x)^{\frac{4}{x}}$ **.**

Let $t = -\frac{1}{x}$. As x goes to zero from below, we see that t goes to $+\infty$. The original problem is the same as

$$\lim_{t\to\infty}\left(1+\frac{-3}{t}\right)^{-4t}.$$

If c is a constant, then $\lim(f^c)$ is equal to $(\lim f)^c$ provided $\lim f$ exists. It follows that our limits is equal to

$$\left(\lim_{t\to\infty}\left(1+\frac{-3}{t}\right)^{t}\right)^{-4}.$$

We learned that for any constant r $\lim\limits_{t\to\infty}\left(1+\frac{r}{t}\right)^t = e^r$; and therefore, our answer is

$$(e^{-3})^{-4} = \boxed{e^{12}.}$$

2. **Find** $\lim\limits_{x\to 0} \frac{1-\cos 3x}{x^2}$ **.**

Multiply top and bottom by $1+\cos 3x$ to see that our problem is equal to

$$\lim_{x\to 0}\frac{1-\cos^2 3x}{x^2(1+\cos 3x)} = \lim_{x\to 0}\frac{\sin^2 3x}{x^2(1+\cos 3x)} = \lim_{x\to 0} 9\frac{\sin 3x}{3x}\frac{\sin 3x}{3x}\frac{1}{(1+\cos 3x)}.$$

We know that $\lim\limits_{t\to 0}\frac{\sin t}{t} = 1$. Apply this fact twice with $3x$ playing the role of t to conclude that the limit is $\boxed{\frac{9}{2}}$.

3. **Find** $\lim_{x\to\infty} \sqrt{x^2-10x} - \sqrt{x^2+4x}$ **.**

Multiply top and bottom by $\sqrt{x^2-10x} + \sqrt{x^2+4x}$ to see that the limit is

$$\lim_{x\to\infty} \frac{x^2-10x-(x^2+4x)}{\sqrt{x^2-10x}+\sqrt{x^2+4x}} = \lim_{x\to\infty} \frac{-14x}{\sqrt{x^2-10x}+\sqrt{x^2+4x}}.$$

Divide top and bottom by x. Keep in mind that when x is positive $x=\sqrt{x^2}$. Our limit is

$$= \lim_{x\to\infty} \frac{-14}{\sqrt{1-\frac{10}{x}}+\sqrt{1+\frac{4}{x}}} = \boxed{-7}.$$

4. **Find a system of parametric equations which parameterizes** $\frac{x^2}{9}+\frac{y^2}{16}=1$ **.**

The easiest parameterization of this ellipse is:

$$\boxed{\begin{cases} x = 3\cos t \\ y = 4\sin t \end{cases}}$$

Notice that if one eliminates the parameter, then the resulting Cartesian equation is $\frac{x^2}{9}+\frac{y^2}{16}=1$. My parameterization starts at $(3,0)$ at $t=0$ and moves counterclockwise. The first time that the object returns to $(3,0)$ is $t=2\pi$.

5. **The position of an object at time** t **is given by**

$$\begin{cases} x = t-1 \\ y = t^2+2. \end{cases}$$

(a) **Eliminate the parameter to find a Cartesian equation for the path of the object.**

$\boxed{y=(x+1)^2+2}$.

(b) **Graph the path of the object.**

See a different page.

(c) **On your graph, mark the position of the object at a few particular values for time.**

See a different page.

6. **Solve** $e^{-2x} - 3e^{-x} = -2$ **.**

Factor $e^{-2x} - 3e^{-x} + 2 = 0$ to get $(e^{-x} - 1)(e^{-x} - 2) = 0$. So, $e^{-x} = 1$ or $e^{-x} = 2$. Take the natural logarithm of each side $-x = \ln 1$ or $-x = \ln 2$. Of course, we know that $\ln 1 = 0$. We conclude that $\boxed{x = 0 \text{ or } x = -\ln 2}$.

7. **Solve** $\ln(4x) - 3\ln(x^2) = \ln 2$ **.**

Exponentiate with base e to see that

$$e^{\ln(4x) - 3\ln(x^2)} = e^{\ln 2}$$

$$e^{\ln(4x)} e^{-3\ln(x^2)} = 2$$

$$\frac{4x}{x^6} = 2$$

Thus $2 = x^5$ and $\boxed{x = \sqrt[5]{2}}$.

8. **Simplify** $\sin(\cos^{-1} x)$ **. Your answer should not contain any Trig functions or inverse Trig functions.**

Draw a right triangle and lable one of the non-right angles as $\cos^{-1} x$. So, the adjacent side has length x and the hypotenuse has length 1 . Use the pythagorean theorem to see that the opposite has length $\sqrt{1 - x^2}$. In other words, $\sin(\cos^{-1} x)$, which is equal to the oposite over the hypotenuse, is $\boxed{\sqrt{1 - x^2}}$.

9. **Find an equation for the family of lines that pass through the intersection of** $5x - 3y + 11 = 0$ **and** $2x - 9y + 7 = 0$ **.**

We first find the intersection. Multiply the first equation by -2 and add this to 5 times the second equation. The result is $-39y + 13 = 0$; so $y = \frac{1}{3}$. Plug this value for y into either equation to see that the x-coordinate of the intersection is $x = -2$. The point of intersection is $(-2, \frac{1}{3})$. Every line has the form $ax + by + c = 0$. We leave a and b arbitrary, but choose c so that $(-2, \frac{1}{3})$ satisfies the equation. So, $-2a + \frac{1}{3}b + c = 0$. In other words, $c = 2a - \frac{1}{3}b$ and our family of lines is: $\boxed{ax + by + 2a - \frac{1}{3}b = 0}$. Notice that the first equation is the member of our family when $a = 5$ and $b = -3$; and the second equation is the member of our family when $a = 2$ and $b = -9$.

10. **Let** $f(x) = \frac{x-2}{x+3}$ **.**

(a) **What is the domain of** f **?**

The domain of f is $\boxed{\text{all real } x \text{ except } -3}$.

(b) **Find a formula for** $f^{-1}(x)$ **.**

Let $y = f^{-1}(x)$. So $f(y) = x$. In other words,

$$\frac{y-2}{y+3} = x.$$

We want to find y . Multiply both sides by $y+3$. The result is the following linear expression in y :

$$y - 2 = x(y+3).$$

We can always solve a linear expression. Get all of the y 's on one side and all of the terms without y on the other side. So,

$$-2 - 3x = y(x-1).$$

or $y = \frac{-2-3x}{x-1}$. Thus,

$$\boxed{f^{-1}(x) = \frac{-2-3x}{x-1}}.$$

(c) **What is the domain of** f^{-1} **?**

The domain of f^{-1} is $\boxed{\text{all real } x \text{ except } 1}$.

(d) **Verify that** $f(f^{-1}(x)) = x$ **for all** x **in the domain of** f^{-1} **.**

Take $x \neq 1$. We see that

$$f(f^{-1}(x)) = f\left(\frac{-2-3x}{x-1}\right) = \frac{\frac{-2-3x}{x-1} - 2}{\frac{-2-3x}{x-1} + 3} = \frac{-2-3x-2(x-1)}{-2-3x+3(x-1)} = \frac{-5x}{-5} = x. \checkmark$$

(e) **Verify that** $f^{-1}(f(x)) = x$ **for all** x **in the domain of** f **.**

Take $x \neq -3$. We see that

$$f^{-1}(f(x)) = f^{-1}\left(\frac{x-2}{x+3}\right) = \frac{-2-3\left(\frac{x-2}{x+3}\right)}{\frac{x-2}{x+3} - 1} = \frac{-2(x+3)-3(x-2)}{x-2-(x+3)} = \frac{-5x}{-5}$$
$$= x. \checkmark$$