Exam 4 141 2000

PRINT Your Name:____________________________

Recitation Time ________ Tu. Th.

There are 10 problems on 5 pages. Each problem is worth 10 points. SHOW your work. $\boxed{CIRCLE}$ your answer. **NO CALCULATORS!** You might find the following formulas to be useful:

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \quad \text{and} \quad \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}.$$

1. STATE both parts of the Fundamental Theorem of Calculus.

Let $f(x)$ be a continuous function for $a \le x \le b$.

(a) If $A(x) = \int_a^x f(t)\,dt$, then $A'(x) = f(x)$.

(b) If $G(x)$ is any antiderivative of $f(x)$, then $\int_a^b f(x)\,dx = G(b) - G(a)$.

2. Find $\int x \sin(x^2+4)dx$. Be sure to check your answer.

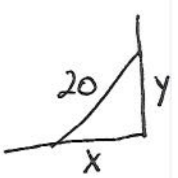
Let $u = x^2+4$    Prob $= \frac{1}{2}\int \sin u \, du = -\frac{1}{2}\cos u + C = -\frac{1}{2}\cos(x^2+4) + C$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$    Check $\frac{d}{dx}(\text{Proposed Answer}) = +\frac{1}{2}(2x)\sin(x^2+4)$ ✓

3. DEFINE the definite integral $\int_a^b f(x)\,dx$. Let $f(x)$ be a function which is defined for $a \le x \le b$. For each partition $P$ of $a \le x \le b$ of the form $a = x_0 \le x_1 \le \cdots \le x_n = b$, let $U_P(f) = M_1(x_1 - x_0) + M_2(x_2 - x_1) + \cdots + M_n(x_n - x_{n-1})$ and $L_P(f) = m_1(x_1 - x_0) + \cdots + m_n(x_n - x_{n-1})$ where $M_i$ is the maximum value of $f(x)$ for $x_{i-1} \le x \le x_i$ and $m_i$ is the minimum value of $f(x)$ for $x_{i-1} \le x \le x_i$ for $1 \le i \le n$. If there is exactly one number between every $L_P(f)$ and every $U_P(f)$ then that number is the definite integral of $f(x)$ for $a \le x \le b$ and that number is denoted $\int_a^b f(x)\,dx$

4. A 20-foot ladder is leaning against a wall. If the bottom of the ladder is pulled along the level pavement directly away from the wall at 5 feet per second, how fast is the top of the ladder moving down the wall when the foot of the ladder is 7 feet from the wall?




$x^2 + y^2 = 400$

$\frac{dx}{dt} = 5$ f/s

Find $\left.\frac{dy}{dt}\right|_{x=7\text{f}}$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$$

$$\left.\frac{dy}{dt}\right|_{x=7} = \frac{-7}{\sqrt{400-49}}\ 5 \text{ f/s}$$

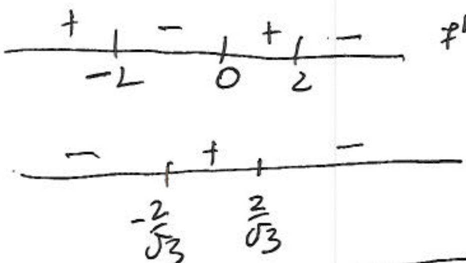
5. Let $y = x\cos^3(4x^2+3) + \sin^4(x)$ . Find $\frac{dy}{dx}$ .

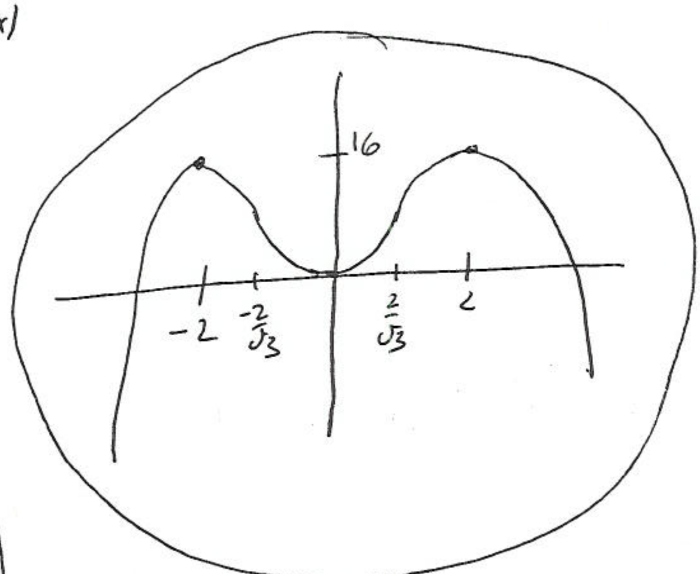
$$\frac{dy}{dx} = -3x\cos^2(4x^2+3)\sin(4x^2+3)\,8x + \cos^3(4x^2+3) + 4\sin^3 x\cos x$$

6. Let $f(x) = 8x^2 - x^4$ . Where is $f(x)$ increasing, decreasing, concave up, and concave down? Find all local maximum points, local minimum points, and points of inflection of $y = f(x)$ . Graph $y = f(x)$ .

$f' = 16x - 4x^3 = 4x(4-x^2) = 4x(2-x)(2+x)$

$f'' = 16 - 12x^2 = 4(4-3x^2) = 4(2-\sqrt{3}x)(2+\sqrt{3}x)$




f is inc. for $x<-2$ also for $0<x<2$

f is dec for $-2<x<0$, also for $2<x$

f is c.d. for $x<-\frac{2}{\sqrt{3}}$ also for $\frac{2}{\sqrt{3}}<x$

f is c.u. for $-\frac{2}{\sqrt{3}}<x<\frac{2}{\sqrt{3}}$

inf. pt at $\left(-\frac{2}{\sqrt{3}}, f\left(-\frac{2}{\sqrt{3}}\right)\right)$ $\left(\frac{2}{\sqrt{3}}, f\left(\frac{2}{\sqrt{3}}\right)\right)$

loc. max $(-2,16)$ $(2,16)$

loc min $(0,0)$

7. Solve the Initial Value Problem $\frac{dy}{dx} = y^4$, $y(1) = \frac{1}{2}$. Be sure to check your answer.

$\int y^{-4}\,dy = \int dx$

$-\frac{y^{-3}}{3} = x + C$

$\frac{1}{y^3} = -3x - 3C$

$\frac{1}{-3x-3C} = y^3$

$\frac{1}{\sqrt[3]{-3x-3C}} = y$

$\frac{1}{\sqrt[3]{-3-3C}} = y(1) = \frac{1}{2}$

$2 = \sqrt[3]{-3-3C}$

$8 = -3-3C$

$11 = -3C$

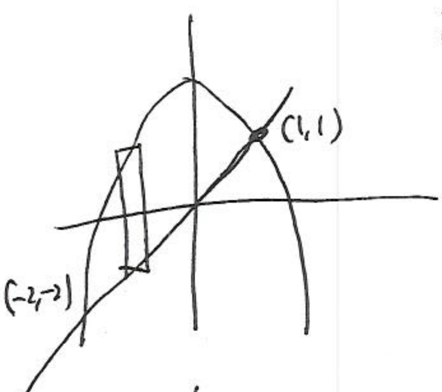
$\therefore\ y = \frac{1}{\sqrt[3]{11-3x}}$

Check $\frac{dy}{dx} = -\frac{1}{3}(11-3x)^{-\frac{4}{3}}(-3)$

$= \frac{1}{(11-3x)^{\frac{4}{3}}} = y^4$ ✓

$y(1) = \frac{1}{\sqrt[3]{11-3}} = \frac{1}{\sqrt[3]{8}} = \frac{1}{2}$ ✓

8. Find the area of the region between $y = 2 - x^2$ and $y = x$.




intersection $x = 2 - x^2$

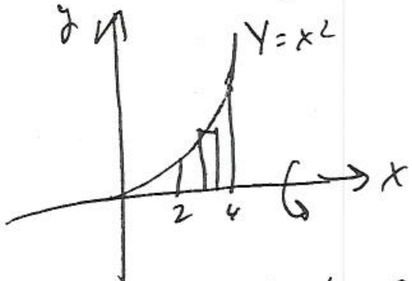
$x^2 + x - 2 = 0$

$(x+2)(x-1) = 0$

$x = -2, 1$

area: $\int_{-2}^{1} 2 - x^2 - x\,dx = 2x - \frac{x^3}{3} - \frac{x^2}{2}\Big]_{-2}^{1} = \left(2 - \frac{1}{3} - \frac{1}{2} - \left(-4 + \frac{8}{3} - \frac{4}{2}\right)\right)$

9. Consider the region bounded by $y = x^2$, $y = 0$, $x = 2$ and $x = 4$. Rotate this region about the $x$-axis. Find the volume of the resulting solid.

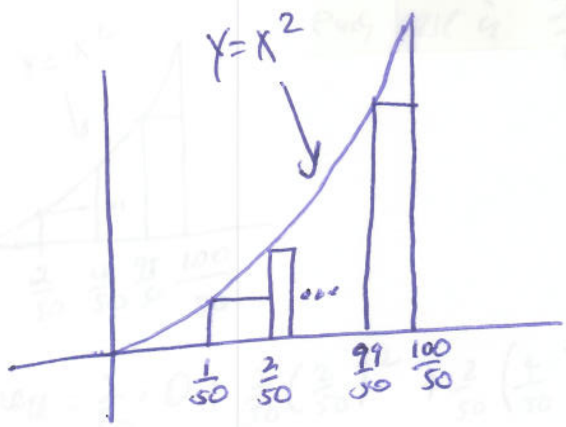



Spin the rectangle. Get a disk of vol

$\pi r^2 t$ where $r = x^2$ $t = dx$

$$Vol = \pi \int_2^4 x^4\,dx = \pi \frac{x^5}{5}\Big]_2^4 = \pi\left(\frac{4^5}{5} - \frac{2^5}{5}\right)$$

10. Consider the region $A$, which is bounded by the $x-$ axis, $y = x^2$, $x = 0$, and $x = 2$. Consider 100 rectangles, all with base $1/50$, which UNDER estimate the area of $A$. How much area is inside the 100 rectangles? (You must answer the question I asked. I expect an exact answer in closed form.)




area inside the boxes is

$$\frac{1}{50}(0) + \frac{1}{50}\left(\frac{1}{50}\right)^2 + \frac{1}{50}\left(\frac{2}{50}\right)^2 + \cdots + \frac{1}{50}\left(\frac{99}{50}\right)^2$$

$$= \left(\frac{1}{50}\right)^3 (1^2 + 2^2 + \cdots + 99^2) = \frac{99(100)(199)}{(50)^3\, 6}$$