1996 Exam 3

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PRINT Your Name:________________________________

There are 8 problems on 5 pages. The exam is worth a total of 100 points. SHOW your work. $\boxed{CIRCLE}$ your answer. **NO CALCULATORS!**

1. (12 points) Let $3x^3y^2 + \sin(2xy^2) = 4y^2 + 9x^2$. Find $\frac{dy}{dx}$.

$$3x^3 2y\frac{dy}{dx} + 9x^2y^2 + \left(4xy\frac{dy}{dx} + 2y^2\right)\cos(2xy^2) = 8y\frac{dy}{dx} + 18x$$

$$\boxed{\frac{dy}{dx} = \frac{18x - 9x^2y^2 - 2y^2\cos(2xy^2)}{6x^3y + 4xy\cos(2xy^2) - 8y}}$$

2. (12 points) Let $y = \sqrt{\sin^4(9x^3 + 4x^2 + 19) + \cos^3 x}$. Find $\frac{dy}{dx}$.

$$\boxed{\frac{dy}{dx} = \frac{4\sin^3(9x^3+4x^2+19)\cos(9x^3+4x^2+19)(27x^2+8x) - 3\cos^2 x\sin x}{2\sqrt{\sin^4(9x^3+4x^2+19)} + \cos^3 x}}$$

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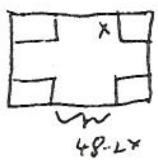
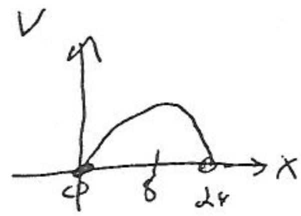
3. (13 points) Find the dimensions of the open box of largest volume that can be made from a square piece of cardboard, 48 inches on each side, by cutting equal squares from the corners and turning up the sides.



$$V = (48-2x)^2 x \qquad 0 \le x \le 24$$

$$V' = 2(48-2x)(-2)x + (48-2x)^2$$
$$= (48-2x)[-4x + 48-2x]$$
$$= [48-2x][-6x+48]$$
$$= [48-2x](-6)(x-8)$$



the box should be 32" × 32" × 8"

4. (13 points) Each edge of a cube is growing at the constant rate of 5 inches per second. How fast is the surface area growing when each edge is 10 inches long?

ℓ = length of each edge

A = surface area

we know $\dfrac{d\ell}{dt} = 5$

$$A = 6\ell^2$$

we want $\left.\dfrac{dA}{dt}\right|_{\ell=10}$

$$\frac{dA}{dt} = 12\ell \frac{d\ell}{dt}$$

$$\left.\frac{dA}{dt}\right|_{\ell=10} = 12(10)5 = 600 \frac{\text{in}^2}{\text{sec}}$$

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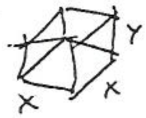
5. (13 points) A rectangular box with a square base is to be constructed to hold 160 cubic yards of water. If the metal top costs ~~five~~ 4 times as much per square yard as the concrete sides and base, what are the most economical dimensions for the box?



Let 1 sq yd of concrete cost $d

then 1 sq yd of metal costs 4d $

$$C = \underbrace{4dx^2}_{\text{top}} + \underbrace{4dxy}_{\text{sides}} + \underbrace{dx^2}_{\text{bottom}}$$

$$160 = x^2 y$$

$$\frac{160}{x^2} = y$$

$$C = 5dx^2 + \frac{640d}{x}$$

$$C' = 10dx - \frac{640d}{x^2}$$

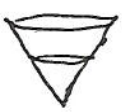
$C' = 0$ when $10d\left[x^3 - 64\right] = 0$

$$x = 4 \quad y = 10$$

the box should be

4 yds by 4 yds by 10 yds

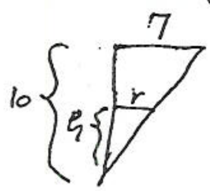
6. (13 points) A student is using a straw to drink from a conical cup, whose axis is vertical, at the rate of 6 cubic inches per second. If the height of the cup is 10 inches and the radius of its opening is 7 inches, how fast is the level of the liquid falling when the depth of the liquid is 3 inches? (Recall that the volume of a cone is $V = \frac{1}{3}\pi r^2 h$.)



V = vol of the liquid

h = ht of the liquid

r = rad of the top of the liquid



$$\frac{h}{10} = \frac{r}{7}$$

$$\frac{7}{10}h = r$$

Know $\frac{dV}{dt} = -6$ in³/s

want $\left.\frac{dh}{dt}\right|_{h=3}$

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \frac{49}{100} h^3$$

$$\frac{dV}{dt} = \frac{\pi 49}{100} h^2 \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{100}{49\pi h^2} \frac{dV}{dt}$$

$$\left.\frac{dh}{dt}\right|_{h=3} = \frac{100}{49\pi(9)}(-6) \text{ in/sec}$$

$$\frac{49}{9} \over 441$$

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7. (12 points) Let $f(x) = -x^3 + 3x^2 - 2$. Where is $f(x)$ increasing, decreasing, concave up, and concave down? What are the local extreme points and points of inflection of $y = f(x)$. Find all vertical and horizontal asymptotes. Graph $y = f(x)$.

$\boxed{\text{No asymptotes}}$

$f' = -3x^2 + 6x = -3x(x-2)$

$f'' = -6x + 6 = -6(x-1)$

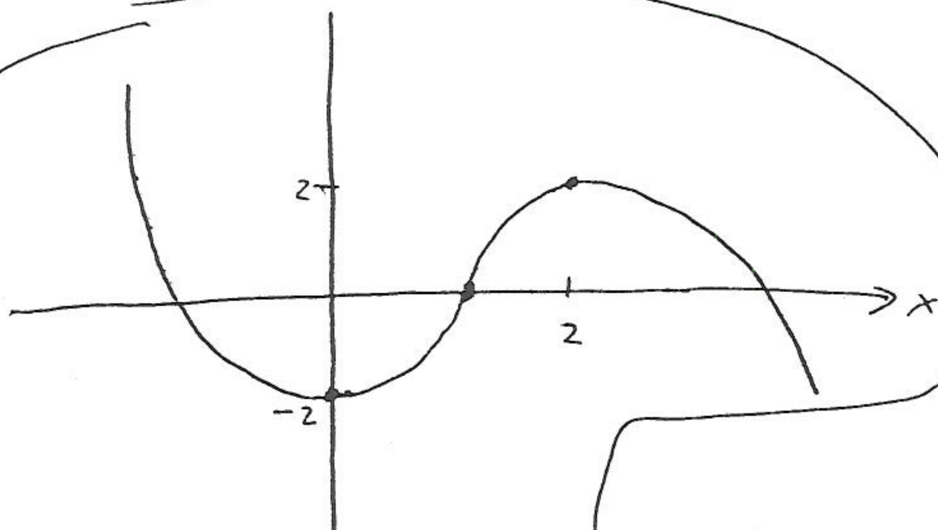
f' neg | 0: $f' = 0$ | f' pos | 2: $f' = 0$ | f' neg

f' pos | 1: $f'' = 0$ | f' neg

$f(1) = 0$

$f(0) = -2$

$f(2) = -8 + 12 - 2 = 2$



f is inc for $0 \le x \le 2$

f is dec for $x < 0$ and $2 < x$

f is c.u. for $x < 1$

f is c.d. for $1 < x$

local min $(0, -2)$

local max $(2, 2)$

Pt. i $(1, 0)$

No asymptotes

8. (12 points) Let $f(x) = \dfrac{1}{1+x^2}$. Where is $f(x)$ increasing, decreasing, concave up, and concave down? What are the local extreme points and points of inflection of $y = f(x)$. Find all vertical and horizontal asymptotes. Graph $y = f(x)$.

$\lim_{x\to\infty} f = 0$ $\quad \lim_{x\to-\infty} f = 0$ $\quad$ so $y=0$ is a horiz. asym.

$$f' = \frac{-2x}{(1+x^2)^2} = -2x(1+x^2)^{-2}$$

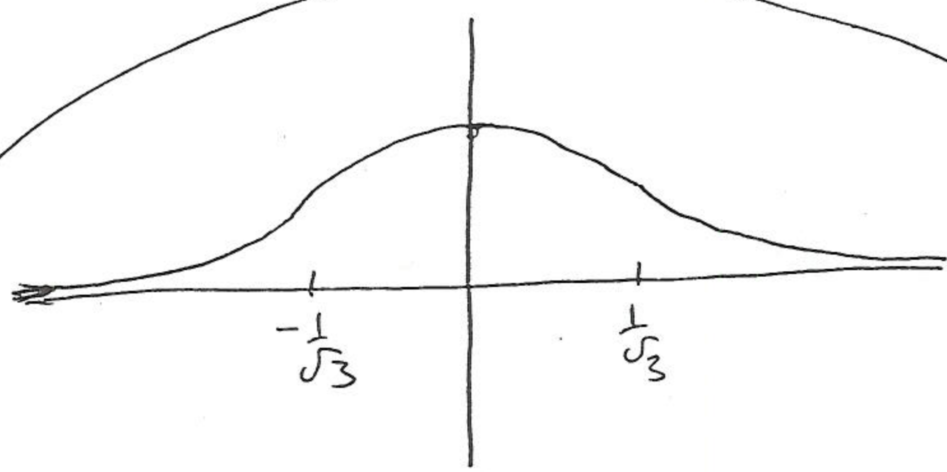
$$f'' = 4x(1+x^2)^{-3}2x - 2(1+x^2)^{-2}$$

$$= \frac{2}{(1+x^2)^3}\left[4x^2 - (1+x^2)\right]$$

$$= \frac{2}{(1+x^2)^3}\left[3x^2-1\right]$$

f' pos $\quad$ $f'=0$ at 0 $\quad$ f' neg

f'' pos $\quad$ $-\frac{1}{\sqrt3}$ $\quad$ f'' neg $\quad$ $\frac{1}{\sqrt3}$ $\quad$ f'' pos



$(0,1)$ local max

$\left(\frac{1}{\sqrt3}, \frac34\right)$ $\left(\frac{-1}{\sqrt3}, \frac34\right)$ points of infl

$y=0$ is a hor. asy $\quad$ No V. asymptotes.

f is inc for $x<0$

f is dec for $0<x$

f is c.u. for $x<-\frac{1}{\sqrt3}$ also for $\frac{1}{\sqrt3}<x$

f is c.d. for $-\frac{1}{\sqrt3}<x<\frac{1}{\sqrt3}$