1995 Exam 3

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PRINT Your Name:______________________________

There are 14 problems on 7 pages. Problems 10 and 12 are worth 8 points each. Each of the other problems is worth 7 points. SHOW your work. $\boxed{CIRCLE}$ your answer. **NO CALCULATORS!**

1. State the Mean Value Theorem. If $f(x)$ is differentiable for $a \le x \le b$, then there exists c, with $a \le c \le b$, such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

2. Let $y = \dfrac{2x}{\sqrt{2x^2+1}} - \sin(2x)$. Find $\frac{dy}{dx}$.

$$y' = -2x\left(\tfrac{1}{2}\right)(2x^2+1)^{-\frac{3}{2}}4x + \frac{2}{\sqrt{2x^2+1}} - 2\cos 2x$$

3. Let $y = (3x^4 + \sqrt{2}x)^5(2x^5 + \cos(3x^2))^6$. Find $\frac{dy}{dx}$.

$$y' = (3x^4+\sqrt{2}x)^5\, 6\,(2x^5+\cos(3x^2))^5\,(10x^4 - 6x\sin 3x^2)$$
$$+ (2x^5+\cos(3x^2))^6\, 5\,(3x^4+\sqrt{2}x)^4\,(12x^3+\sqrt{2})$$

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4. Let $4xy^2 + \cos(x^2y) = 3y^2 + 8x^2$. Find $\frac{dy}{dx}$.

$8xy\frac{dy}{dx} + 4y^2 - \sin(x^2y)\left[x^2\frac{dy}{dx} + 2xy\right] = 6y\frac{dy}{dx} + 16x$

$$\frac{dy}{dx} = \frac{16x + 2xy\sin(x^2y) - 4y^2}{8xy - x^2\sin(x^2y) - 6y}$$

5. Let $y = \sqrt{\cos^3(4x^2 + 3x + 19) + \sin^4(x)}$. Find $\frac{dy}{dx}$.

$$y' = \frac{-3\cos^2(4x^2+3x+19)\sin(4x^2+3x+19)(8x+3) + 4\sin^3 x\cos x}{2\sqrt{\cos^3(4x^2+3x+19) + \sin^4(x)}}$$

6. Find $\int 2x^2 + \sin(2x)dx$.

$= \frac{2}{3}x^3 - \frac{1}{2}\cos 2x + C$

7. Find $\int \frac{x}{\sqrt{2x^2+1}}\,dx$. $= \frac{1}{2}\sqrt{2x^2+1} + C$

8. Let $f(x) = x^3 - 3x$. Where is $f(x)$ increasing, decreasing, concave up, and concave down? What are the local extreme points and points of inflection of $y = f(x)$. Find all vertical and horizontal asymptotes. Graph $y = f(x)$.

$f' = 3x^2 - 3 = 3(x^2-1)$

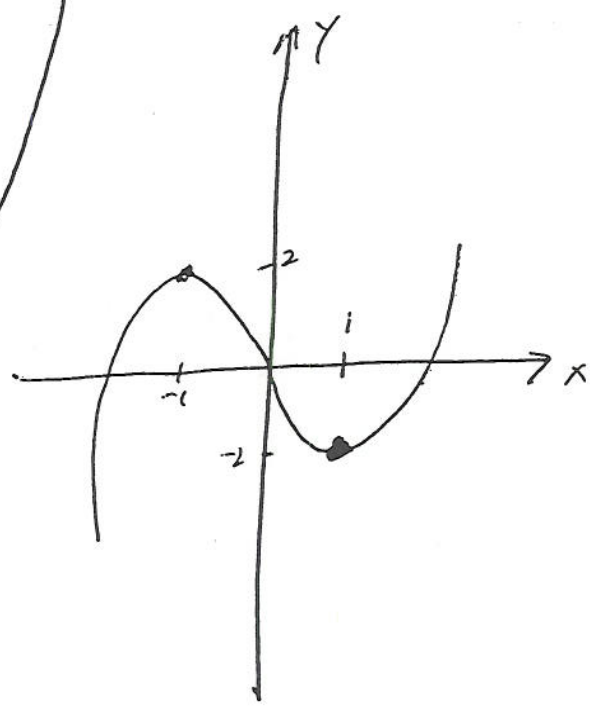
$f'' = 6x$

f': $+$ | -1 | $-$ | 1 | $+$ → x

f'': $-$ | 0 | $+$ → x

f is inc for $x<-1$, also for $1<x$
f is dec for $-1<x<1$
f is c.d for $x<0$
f is c.u. for $0<x$

$(-1,2)$ is a local max
$(1,-2)$ is a local min
$(0,0)$ is a p of i
No v or h asy



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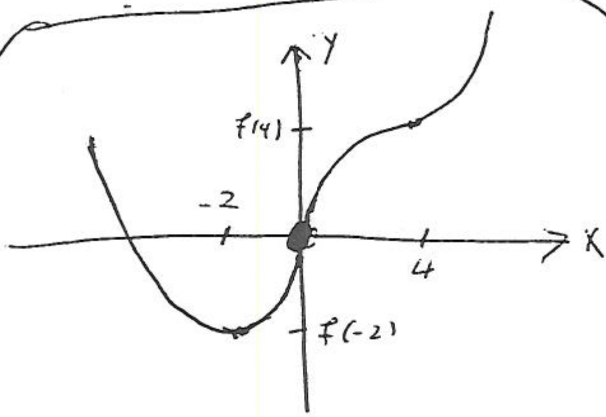
9. Let $f(x) = 8x^{1/3} + x^{4/3}$. Where is $f(x)$ increasing, decreasing, concave up, and concave down? What are the local extreme points and points of inflection of $y = f(x)$. Find all vertical and horizontal asymptotes. Graph $y = f(x)$.

$$f' = \frac{8}{3}x^{-\frac{2}{3}} + \frac{4}{3}x^{\frac{1}{3}} = \frac{4}{3}x^{-\frac{2}{3}}(2 + x)$$

$$f'' = -\frac{16}{9}x^{-\frac{5}{3}} + \frac{4}{9}x^{-\frac{2}{3}} = \frac{4}{9}x^{-\frac{5}{3}}(-4 + x)$$

f': $- - -$ (−2) $+ + +$ (0) $+ + + + + + + +$ $\to x$

f'': $+ + +$ (0) $- - -$ (4) $+ + \ldots$ $\to x$



$(-2, f(-2))$ is a local min

$(0, 0)$ is a poi

$(4, f(4))$ is a poi

dec for $x < -2$

inc for $-2 < x$

cu for $x < 0$, also for $4 < x$

cd for $0 < x < 4$

no v. or h asy

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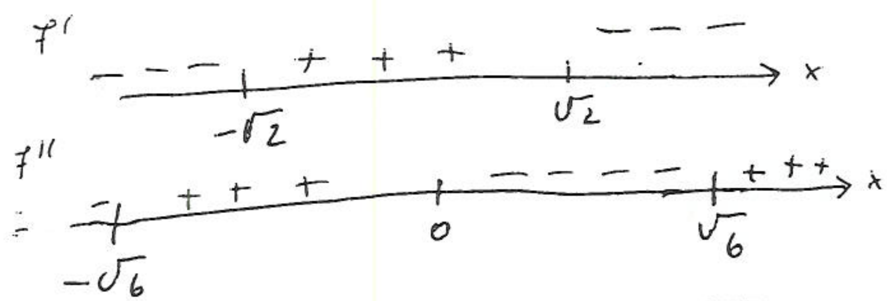
10. Let $f(x) = \dfrac{4x}{x^2+2}$. Where is $f(x)$ increasing, decreasing, concave up, and concave down? What are the local extreme points and points of inflection of $y = f(x)$. Find all vertical and horizontal asymptotes. Graph $y = f(x)$.

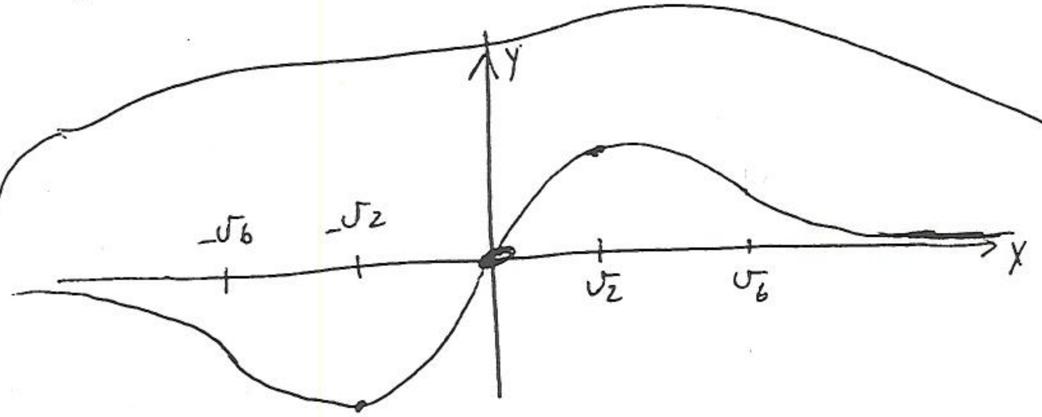
$\lim_{x\to\infty} = 0 \quad \lim_{x\to-\infty} = 0$ h asy at $y=0$ no V. asy

$f'(x) = \dfrac{(x^2+2)4 - 4x\,2x}{(x^2+2)^2} = \dfrac{-4x^2+8}{(x^2+2)^2} = \dfrac{-4(x^2-2)}{(x^2+2)^2} = (-4x^2+8)(x^2+2)^{-2}$

$f''(x) = (-4x^2+8)(-2)(x^2+2)^{-3}\,2x + (x^2+2)^{-2}(-8x)$

$= \dfrac{(-4x)[-4x^2+8+2(x^2+2)]}{(x^2+2)^3} = \dfrac{-4x[-2x^2+12]}{(x^2+2)^3} = \dfrac{8x(x^2-6)}{(x^2+2)^3}$





$(\sqrt{2}, f(\sqrt{2}))$ is a rel. max

$(-\sqrt{2}, f(-\sqrt{2}))$ is a rel min

$(-\sqrt{6}, f(-\sqrt{6}))$, $(0,0)$ $(\sqrt{6}, f(\sqrt{6}))$ are poi

f is dec for $x < -\sqrt{2}$ also for $\sqrt{2} < x$

f is inc for $-\sqrt{2} < x < \sqrt{2}$

f is c.u. for $-\sqrt{6} < x < 0$, also for $\sqrt{6} < x$

f is c.d. for $x < -\sqrt{6}$, also for $0 < x < \sqrt{6}$

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11. A cube is growing at the constant rate of 1000 cubic inches per second. How fast is the surface area growing when each edge is 5 inches long?

$1000 = \frac{dV}{dt}$ Find $\left.\frac{dA}{dt}\right|_{\ell=5}$

$V=\ell^3$

$A = 6\ell^2$

$\frac{dV}{dt} = 3\ell^2 \frac{d\ell}{dt}$

$\frac{1}{3\ell^2}\frac{dV}{dt} = \frac{d\ell}{dt}$

$\frac{dA}{dt} = 12\ell\frac{d\ell}{dt} = \frac{12\ell}{3\ell^2}\frac{dV}{dt} = \frac{4}{\ell}\frac{dV}{dt}$

$\left.\frac{dA}{dt}\right|_{\ell=5} = \frac{4}{5}\, 1000 = 800$ in^2/sec.

12. A student is using a straw to drink from a conical cup, whose axis is vertical, at the rate of 5 cubic inches per second. If the height of the cup is 9 inches and the radius of its opening is 4 inches, how fast is the level of the liquid falling when the depth of the liquid is 2 inches? (Recall that the volume of a cone is $V = \frac{1}{3}\pi r^2 h$.)

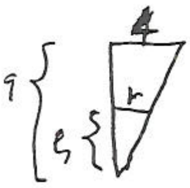
V = vol of the liquid at time t

h = ht of the liquid

r = radius of the liquid

$\frac{dV}{dt} = 5$

want $\left.\frac{dh}{dt}\right|_{h=2}$



$\frac{r}{h} = \frac{4}{9}$

$r = \frac{4}{9}h$

$V = \frac{1}{3}\pi r^2 h$

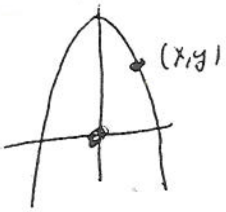
$V = \frac{1}{3}\frac{16}{81}\pi h^3$

$\frac{dV}{dt} = \frac{16}{81}\pi h^2 \frac{dh}{dt}$

$\frac{81}{16\pi h^2}\frac{dV}{dt} = \frac{dh}{dt}$

$\frac{(81)5}{16\pi 4}$ in/sec $= \left.\frac{dh}{dt}\right|_{h=2}$

13. Find the points on the curve $y = 10 - x^2$ which are closest to the point $(0,0)$.



Let D = distance from (x,y) to (0,0)

$D = \sqrt{x^2+y^2} = \sqrt{10-y+y^2}$ $\quad \frac{dD}{dy} = \frac{-1+2y}{2\sqrt{10-y+y^2}}$ $\quad \frac{dD}{dy} = 0$ when $y = \frac{1}{2}$

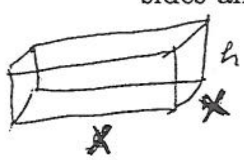
D' — — — — — | + + + + + → y

$\frac{1}{2}$

So minimum distance occurs $y = \frac{1}{2}$ and $x = \pm\sqrt{10-\frac{1}{2}} = \pm\sqrt{\frac{19}{2}}$

$\left(\pm\sqrt{\frac{19}{2}}, \frac{1}{2}\right)$

14. A cistern with a square base is to be constructed to hold 12,000 cubic feet of water. If the metal top costs twice as much per square foot as the concrete sides and base, what are the most economical dimensions for the cistern?



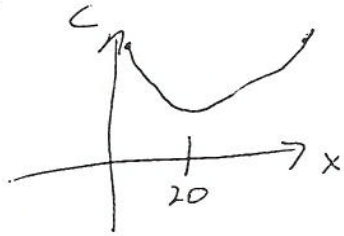
$12000 = x^2 h$ $\qquad$ sides cost \$d / ft^2

$C = d\,(\underbrace{x^2}_{\text{bottom}} + \underbrace{4xh}_{\text{sides}} + \underbrace{2x^2}_{\text{top}})$

$C = d\,(3x^2 + 4xh)$

$= d\left(3x^2 + 4x\,\frac{12000}{x^2}\right)$

$= d\left(3x^2 + \frac{48{,}000}{x}\right)$



$C' = d\left(6x - \frac{48{,}000}{x^2}\right)$

$= d\left(\frac{6x^3 - 48{,}000}{x^2}\right)$

$= 6d\,\frac{x^3 - 8{,}000}{x^2}$

C' — — — | + + + → x

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$C' = 0$ when $x = 20$ ft, $h = 30$ ft ⟸ Min cost