Math 141, Exam 1, Fall 2005 Solutions

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.; although, by using enough paper, you can do the problems in any order that suits you.

The exam is worth a total of 100 points. SHOW your work. Make your work be coherent and clear. Write in complete sentences whenever this is possible. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible. **No Calculators.**

I will post the solutions on my website a few hours after the exam is finished.

1. **(20 points) Graph $y = x^2$, $y = (x-1)^2$, and $y = x^2 + 1$.**

The first graph is the standard parabola with vertex at the origin and opening up. The second graph is obtained by shifting the first graph to the right by 1. The third graph is obtained by shifting the first graph up by 1. The pictures appear on a different piece of paper.

2. **(20 points) Graph $y = x^{1/3}$, $y = x^{2/3}$, and $y^2 = x^{1/3}$.**

In the first quadrant all three graphs are increasing and concave down. (That is, as x grows, y also grows; however, the growth of y is much slower than the growth of x.) The first graph is symmetric about the origin. The second graph is symmetric about the y-axis. The third graph is symmetric across the x-axis. The pictures appear on a different piece of paper.

3. **(10 points) Find all lines through $(6,-1)$ for which the product of the x and y intercepts is 3.**

The lines in question can not be vertical or horizonal. So, each such line has the form $y = mx + b$. The y intercept of this line occurs when $x = 0$; so, it is b. The x intercept occurs when $y = 0$; so it is $\frac{-b}{m}$. Keep in mind that the line is not horizontal; so $m \neq 0$. Our parameters m and b must satisfy two conditions: $b(\frac{-b}{m}) = 3$ and $-1 = m(6) + b$. The first condition tells us that $\frac{-b^2}{3} = m$. The second condition tells us that $-1 = (\frac{-b^2}{3})6 + b$. In other words, $2b^2 - b - 1 = 0$. We factor this equation to get $(2b+1)(b-1) = 0$. So, $b = \frac{-1}{2}$ or $b = 1$. When $b = \frac{-1}{2}$, then $m = \frac{-1}{12}$. When $b = 1$, then $\frac{-1}{3} = m$. Our answer is:

$$\boxed{y = \frac{-x}{12} - \frac{1}{2} \ \&\ y = \frac{-x}{3} + 1.}$$

Check: Notice that $(6,-1)$ is on both lines. The y intercept of the first line is $-\frac{1}{2}$; the x intercept is -6; the product of the intercepts is 3. The y intercept of the second line is 1; the x intercept is 3; the product of the intercepts is 3.

4. **(10 points) Compute** $\sin(\cos^{-1}(2/3) + \cos^{-1}(1/3))$ **.**

We know that

$$\sin(\theta + \varphi) = \sin(\theta)\cos(\varphi) + \cos(\theta)\sin(\varphi).$$

It follows that

$$\sin(\cos^{-1}(2/3) + \cos^{-1}(1/3))$$
$$= \sin(\cos^{-1}(2/3))\cos(\cos^{-1}(1/3)) + \cos(\cos^{-1}(2/3))\sin(\cos^{-1}(1/3)).$$

It is clear that $\cos(\cos^{-1}(1/3)) = \frac{1}{3}$ and $\cos(\cos^{-1}(\frac{2}{3})) = \frac{2}{3}$. To compute $\sin(\cos^{-1}(2/3))$, draw a triangle with ADJ equal to 2 and HYP equal to 3. It follows that OP is $\sqrt{5}$ and $\sin(\cos^{-1}(2/3)) = \frac{\sqrt{5}}{3}$. To compute $\sin(\cos^{-1}(\frac{1}{3}))$, draw a triangle with ADJ equal to 1 and HYP equal to 3. It follows that OP is $\sqrt{8}$ and $\sin(\cos^{-1}(\frac{1}{3})) = \frac{\sqrt{8}}{3}$. We now know that

$$\sin(\cos^{-1}(2/3) + \cos^{-1}(\frac{1}{3}))$$

$$= \sin(\cos^{-1}(2/3))\cos(\cos^{-1}(\frac{1}{3})) + \cos(\cos^{-1}(2/3))\sin(\cos^{-1}(\frac{1}{3}))$$

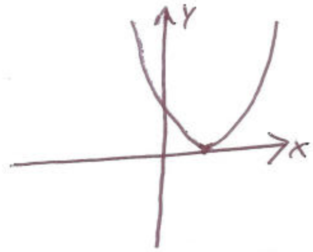
$$= \boxed{\frac{\sqrt{5}}{3}\frac{1}{3} + \frac{2}{3}\frac{\sqrt{8}}{3}}.$$

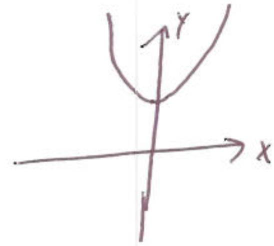
5. **(10 points) Solve** $1 + 3\log_2 x = \log_2(3x)$ **.**

6. **(20 points) Let** $f(x) = x - 5x^2$ **for** $x \le \frac{1}{10}$ **.**
(a)Find a formula for $f^{-1}(x)$ **.**
(b)What is the domain of $f^{-1}(x)$ **?**
(c)Verify that $f(f^{-1}(x)) = x$ **for all** x **in the domain of** f^{-1} **.**
(d)Verify that $f^{-1}(f(x)) = x$ **for all** x **in the domain of** f **.**

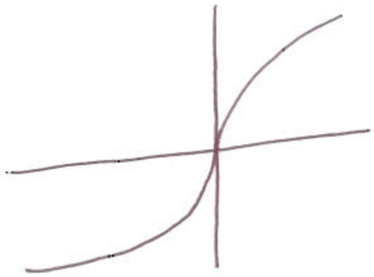
7. **(10 points) An open box is to be constructed from a rectangular sheet of metal, 8 inches by 15 inches, by cutting out squares with sides of length** x **from each corner and bending up the sides. Express the volume** V **as a function of** x **.**

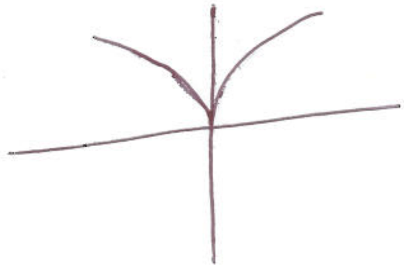
1 $\quad y = x^2 \qquad y = (x-1)^2 \qquad y = x^2 + 1$ ③





② $\quad y = x^{\frac{1}{3}} \qquad y = x^{\frac{2}{3}} \qquad y^2 = x^{\frac{1}{3}}$





⑤ Solve $1 + 3\log_2 X = \log_2(3X)$

Exponentiate base 2

$$2 \cdot X^3 = 3X$$

We know $X > 0$ because X is in the domain of $\log_2$

So we may divide by $2X$. (since $X \neq 0$)

$$X^2 = \frac{3}{2}$$

$$X = \sqrt{\frac{3}{2}} \qquad (\text{again } X > 0)$$

check plug $x = o$ into the equation

$$1 + 3\log_2\sqrt{\tfrac{3}{2}} - \log_2\left(3\sqrt{\tfrac{3}{2}}\right) = 1 + 3\left(\tfrac{1}{2}\cancel{\log_2 3} - \tfrac{1}{2}\log_2 2\right) - \left(\cancel{\log_2 3^{\frac{3}{2}}} - \tfrac{1}{2}\log_2 2\right)$$

$$= 1 - \frac{3}{2} + \frac{1}{2} = 0 \checkmark$$

6. $f(x) = x - 5x^2$ for $x \le \frac{1}{10}$

a) find $f^{-1}(x)$.

Let $y = f^{-1}(x)$.

so $f(y) = x$ & $y \le \frac{1}{10}$

so $y - 5y^2 = x$

$0 = 5y^2 - y + x$

The quadratic formula says if $0 = ay^2 + by + c$, then

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For us $y = \dfrac{1 \pm \sqrt{1 - 20x}}{10}$

Our $y \le \frac{1}{10}$ always so $y = \frac{1}{10} - \dfrac{\sqrt{1-20x}}{10}$

so $\boxed{f^{-1}(x) = \frac{1}{10} - \dfrac{\sqrt{1-20x}}{10}}$ 9

b) We need $1 - 20x \ge 0$ so $1 \ge 20x$ $\boxed{\frac{1}{20} \ge x}$ ← The domain of $f^{-1}(x)$.

c) Take $x \le \frac{1}{20}$.

We calculate

$$f(f^{-1}(x)) = f\left(\frac{1}{10} - \frac{\sqrt{1-20x}}{10}\right)$$

$$= \frac{1}{10} - \frac{\sqrt{1-20x}}{10} - 5\left(\frac{1}{10} - \frac{\sqrt{1-20x}}{10}\right)^2$$

$$= \frac{1}{10} - \cancel{\frac{\sqrt{1-20x}}{10}} - 5\left(\frac{1}{100} - \cancel{\frac{2\sqrt{1-20x}}{100}} + \frac{1-20x}{100}\right)$$

$$= \frac{1}{10} - \frac{1}{20} - \frac{1}{20}(1 - 20x) = \frac{1}{10} - \frac{1}{20} - \frac{1}{20} + x = x \checkmark$$

(d) Take $x \le \frac{1}{10}$. We see that

$$f^{-1}(f(x)) = f^{-1}(x - 5x^2) = \frac{1}{10}\left(1 - \sqrt{1 - 20(x - 5x^2)}\right)$$

$$= \frac{1}{10}\left(1 - \sqrt{1 - 20x + 100x^2}\right)$$

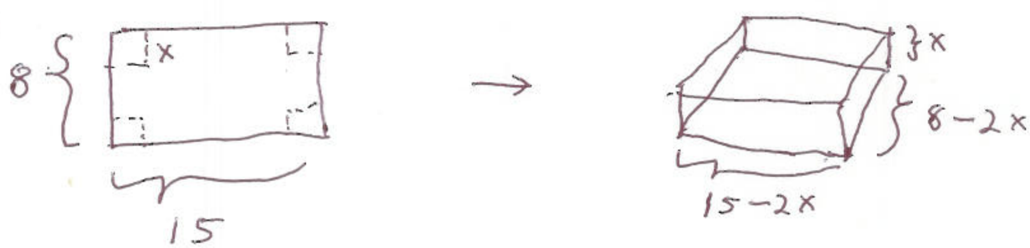
$$= \frac{1}{10}\left(1 - \sqrt{(1 - 10x)^2}\right)$$

$$= \frac{1}{10}\left(1 - |1 - 10x|\right)$$ we have $x \le \frac{1}{10}$ so $0 \le 1 - 10x$

$$= \frac{1}{10}\left(1 - (1 - 10x)\right)$$

$$= \frac{1}{10}\left(1 - 1 + 10x\right) = \frac{10x}{10} = x \checkmark$$

(7)



V = the product of the 3 sides

$$V = (15 - 2x)(8 - 2x)\,x$$