IF THE SOCLE FITS

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The following Lemma, although easy to prove, provides a surprisingly useful way to establish that two ideals are equal.

Lemma 1.1. *Let $R = \oplus_{i \geq 0} R_i$ be a noetherian graded $k-$algebra where R_0 is the field k. Suppose that $K \subseteq J$ are homogeneous ideals of R with R/K an artinian ring. If*

$$\dim_k \operatorname{Socle}(R/K)_\ell \leq \dim_k \operatorname{Socle}(R/J)_\ell$$

for all $\ell \geq 0$, then $K = J$.

In section one we give two versions of our proof of Lemma 1.1. The first version deals directly with the elements of the ring R. The second version is homological in nature. The rest of the paper is devoted to two applications of Lemma 1.1. In section three we give a new proof that the maximal minors of a generic matrix generate a perfect ideal. In section four we compute the generators of a generic residual intersection of a generic grade three Gorenstein ideal. That is, we compute the generators of $J = (\mathfrak{A} : I)$, where I is a generic grade three Gorenstein ideal and $\mathfrak{A}$ is an ideal generated by generic linear combinations of the generators of I. It is not difficult to find some elements in $(\mathfrak{A} : I)$. The difficulty occurs in proving that one has found all of J. In section four we identify our candidate $K \subseteq J$, and then we apply Lemma 1.1 in order to show that $K = J$. Numerical information about the socle of a zero dimensional specialization of R/K requires explicit calculations. These calculations are contained in section five. Numerical information about the socle of a zero dimensional specialization of R/J follows from a general theory. In section two we record the back twists in the minimal homogeneous resolution of R/J in terms of the degrees of the generators of I and $\mathfrak{A}$ where $J = (\mathfrak{A} : I)$ is an arbitrary residual intersection. This part of the paper amounts to interpreting some of the theorems from [12] in a graded context.

The majority of the paper is concerned with applying Lemma 1.1 in order to compute the generators of particular residual intersections. See [1], [11], or [12] for information about the history and significance of residual intersections. The notion of residual intersection is a generalization of linkage. The first theorem about linkage, [16, Proposition 2.6], states that if I is a perfect ideal which is linked to J over $\mathfrak{A}$ in the ring R, then the generators of J and a resolution of R/J may be computed once one knows the generators of $\mathfrak{A}$ and a finite free resolution of R/I. A comparable result about general residual intersections is not yet available. For a summary of the progress that has been made in this direction the reader should consult [3], [13], or [14]. Suffice it to say that the first successful calculation of the generators of the residual intersection J, of a grade three Gorenstein ideal, was made using the arguments of the present paper. In the mean time, the generators of J have also been calculated in [14]. The calculation of J that we give in this paper is completely independent from, and significantly shorter than, the calculation in [14]. Of course, [14] contains many things in addition to a calculation of the generators of J: the quotient R/J is resolved, "half" of the divisor class group of R/J is resolved, and the powers of a grade three Gorenstein ideal are resolved. Furthermore, the ideals J in [14] are treated in a more general context than they are treated in the present paper.

Section 1. Proof of the socle lemma.

First proof of Lemma 1.1. We begin by proving that the natural map

$$\Phi_\ell\colon \operatorname{Socle}(R/K)_\ell \to \operatorname{Socle}(R/J)_\ell$$

is an injection for all ℓ. By induction, we may assume that Φ_m is an injection for all $m > \ell$. If Φ_ℓ is not injective, then the hypothesis guarantees that it is not surjective either. Thus, there is an element $y \in R_\ell$ so that $yR_+ \subseteq J$, but $yR_+ \nsubseteq K$. Select a homogeneous element $z \in R_+$ to have the largest degree among all of the elements with the property that $yz \notin K$. Notice that $yzR_+ \subseteq K$ by the maximality of the degree of z. Hence, yz represents a non-trivial element of the kernel of Φ_m for some $m > \ell$. The map Φ_m is injective by the induction hypothesis; this contradiction implies that Φ_ℓ is also injective.

We finish the argument by assuming that $K \neq J$. Select a homogeneous element x of largest degree with $x \in J$, but $x \notin K$. It follows that x represents a non-trivial element in the socle of R/K. Furthermore, this element lies in the kernel of the map $\operatorname{Socle}(R/K) \to \operatorname{Socle}(R/J)$, which is impossible by the injectivity of this map. $\square$

Our second proof of Lemma 1.1 is derived from the following, apparently more general, result.

Proposition 1.2. *Let $S = k[x_1, \dots, x_n]$ be a positively graded polynomial ring over a field, let $A \subseteq B$ be homogeneous perfect ideals of S of the same grade c, and let $\mathbb{F}$ and $\mathbb{G}$ be the minimal homogeneous $S-$resolutions of S/A and S/B, respectively. Suppose that $F_c = \oplus_{i=1}^m S(-d_i)^{e_i}$ and $G_c = \oplus_{i=1}^m S(-d_i)^{f_i}$, where $d_1 < d_2 < \dots < d_m$. If $0 \le e_i \le f_i$ for all i, then $A = B$.*

Proof. Let $\pi\colon S/A \to S/B$ be the natural map. Recall that

$$\pi^*\colon \operatorname{Ext}_S^c\left((S/B), S\right) \to \operatorname{Ext}_S^c\left((S/A), S\right)$$

is an injection. Indeed, $\operatorname{Ext}_S^{c-1}\left((B/A), S\right)$ is equal to zero because the annihilator of B/A contains a regular $S-$sequence of length c. The map π induces the commutative diagram:

$$\begin{array}{ccccccc} \dots & \longrightarrow & \oplus S(d_i)^{f_i} & \xrightarrow{\varepsilon} & \operatorname{Ext}_S^c((S/B), S) & \longrightarrow & 0 \\ & & {\scriptstyle M}\big\downarrow & & {\scriptstyle \pi^*}\big\downarrow & & \\ \dots & \longrightarrow & \oplus S(d_i)^{e_i} & \longrightarrow & \operatorname{Ext}_S^c((S/A), S) & \longrightarrow & 0. \end{array}$$

We view the map M as a matrix of maps (M_{ij}) where M_{ij} is a map $S(d_j)^{f_j} \to$

$S(d_i)^{e_i}$. Degree considerations show that $M_{ij} = 0$ for $i < j$, and that every entry of M_{ii} is an element of the field k.

We claim that M is an isomorphism. Since M is a lower triangular matrix of maps, it suffices to show that each M_{ii} is an isomorphism. Furthermore, since M_{ii} is a linear transformation from a vector space of dimension f_i to a vector space of dimension e_i, and $e_i \le f_i$ by hypothesis, it suffices to prove that each map M_{ii} is injective. Suppose, by induction, that M_{jj} is an isomorphism for $j > i$; but that M_{ii} is not an injection. In this case, there is an element $x_i \in S(d_i)^{f_i}$, with $x_i \notin S_+\left(S(d_i)^{f_i}\right)$, such that $M_{ii}(x_i) = 0$. Since M_{jj} is surjective for $i+1 \le j \le m$, there exists $x_j \in S(+d_j)^{f_j}$ such that $M\left(\sum_{j=i}^m x_j\right) = 0$. On the other hand, the map π^* is injective; so

$$\ker(M) \subseteq \ker(\varepsilon) \subseteq S_+\left(\oplus S(d_i)^{f_i}\right).$$

This contradiction proves that M is an isomorphism.

It follows that π^* is an isomorphism. The property of perfection guarantees that

$$(\pi^*)^* : \operatorname{Ext}^c_S\left(\operatorname{Ext}^c_S\left((S/A), S\right),\ S\right) \to \operatorname{Ext}^c_S\left(\operatorname{Ext}^c_S\left((S/B), S\right),\ S\right)$$

is exactly the same as $\pi\colon S/A \to S/B$. Thus, π is an isomorphism of $S-$modules, and $A = B$. $\square$

The connection between the back twists in a minimal resolution and the socle type of an artinian ring is well-known. See, for example, [7, 4.g] or [8, Proposition 3.1]. We have included the following proof for the sake of completeness.

Lemma 1.3. *Let $S = k[x_1, x_2, \ldots, x_n]$ be a positively graded polynomial ring over a field k, S/A be a graded artinian quotient of S, and $\mathbb{F}$ be the minimal homogeneous resolution of S/A by free $S-$modules. If $F_n = \oplus_{i=1}^r S(-d_i)$, then there is a (homogeneous degree zero) isomorphism* $\operatorname{Socle}(S/A) \cong \oplus_{i=1}^r k(-(d_i - \Delta))$ *of graded vector spaces, where Δ represents the sum $\sum_{i=1}^n \deg x_i$.*

Proof. The graded object $\operatorname{Tor}^S_n(S/A, k)$ may be computed as $H_n(\mathbb{F} \otimes k) = \oplus_{i=1}^r k(-d_i)$. It may also be computed as $H_n\left(\mathbb{K} \otimes (S/A)\right) = \operatorname{Socle}(S/A)(-\Delta)$, where $\mathbb{K}$ is the Koszul complex on $x_1, x_2, \ldots, x_n$. $\square$

Second proof of Lemma 1.1. The ring R is a quotient of a positively graded polynomial ring $S = k[x_1, x_2, \ldots, x_n]$ and there ideals $A \subseteq B$ in S such that $S/A = R/K$ and $S/B = R/J$. In this case, A and B are both primary to the irrelevant maximal ideal of S. As such, they are perfect ideals of grade n. The result follows from Proposition 1.2 by way of Lemma 1.3. $\square$

Section 2. The back twists in the resolution of a residual intersection.

Let $J = (\mathfrak{A} : I)$ be an $f-$residual intersection in a Gorenstein local ring R. Assume that I is strongly Cohen-Macaulay (i.e., all homology modules of the Koszul complex on a generating set of I are Cohen-Macaulay, cf. [11]), has grade c (with $c \le f$), and satisfies the condition G_∞ (i.e., for every prime ideal P containing I, the number of generators of I_P is at most the height of P, cf. [1]). Huneke and Ulrich [12, Theorem 5.1] have proved that R/J is a Cohen-Macaulay ring whose canonical module is isomorphic to the symmetric power $\operatorname{Sym}_{f-c+1}(I/\mathfrak{A})$. In this section we obtain the following graded version of that result.

Proposition 2.1. *Let $R = k[x_1, x_2, \ldots, x_t]$ be a positively graded polynomial ring over a field k, and let $J = (\mathfrak{A} : I)$ be an $f-$residual intersection of homogeneous ideals in R. Assume that I is strongly Cohen-Macaulay, has grade c (with $c \le f$), and satisfies the condition G_∞. Let*

$$\oplus_{i=1}^{g} R(-m_i) \to R \to R/I \to 0$$
$$\oplus_{j=1}^{f} R(-d_j) \to R \to R/\mathfrak{A} \to 0$$

be minimal homogeneous presentations. Let D represent the sum $\sum_{j=1}^{f} d_j$, and let $\mathcal{I}$ represent the set

$$\{(i) \mid (i) \text{ is a sequence of the form } i_1, \ldots, i_{f-c+1} \text{ with } 1 \le i_1 \le i_2 \le \cdots \le i_{f-c+1} \le g\}.$$

For each (i) in $\mathcal{I}$, let $M_{(i)}$ be the positive integer $m_{i_1} + \ldots + m_{i_{f-c+1}}$. If $m_i < d_j$ for all i and j, then J is a perfect ideal of grade f and the final non-zero module in the minimal homogeneous resolution of R/J has the form

$$\oplus_{(i)\in\mathcal{I}} R\left(-\left(D - M_{(i)}\right)\right).$$

In the course of proving Proposition 2.1, it is necessary to view the canonical module of $A = R/J$ as a graded module whose grading depends on A but not on the presentation

$$0 \to J \to R \to A \to 0.$$

Any such grading convention will work. We use the following standard convention.

Convention 2.2. *Let $A = \oplus_{i\ge0} A_i$ be a graded Cohen-Macaulay $k-$algebra, where A_0 is the field k. Suppose that $R = k[x_1, x_2, \ldots, x_t]$ is a positively graded polynomial ring which maps onto A. If*

$$0 \to \oplus_i R(-n_i) \to \cdots \to R \to A \to 0$$

is a minimal homogeneous $R-$resolution of A, and Δ represents the sum $\sum_{i=1}^{t} \deg x_i$, then

$$0 \to R(-\Delta) \to \cdots \to \oplus_i R\left(-\left(\Delta - n_i\right)\right) \to \omega_A \to 0$$

is a minimal homogeneous resolution of the canonical module ω_A of A.

The next result is a graded version of [12, Lemma 2.1].

Lemma 2.3. *Let $A = \oplus_{i\ge0} A_i$ be a graded Cohen-Macaulay $k-$algebra where A_0 is the field k, and let $I \subset A$ be a homogeneous strongly Cohen-Macaulay ideal of grade c. Let $\underline{\alpha}$ be a homogeneous regular $A-$sequence $\alpha_1, \ldots, \alpha_c$. If $(\underline{\alpha}) \subsetneqq I$, and J is the ideal $((\underline{\alpha}) : I)$, then A/J is a Cohen-Macaulay ring and there is a (homogeneous degree zero) isomorphism*

$$\omega_{A/J} \cong \left(\frac{I\omega_A}{(\underline{\alpha})\omega_A}\right)\left(\sum_{i=1}^{c} \deg \alpha_i\right)$$

of graded $A-$modules.

Proof. Let $R = k[x_1, \ldots, x_t]$ be a positively graded polynomial ring which maps onto A, $\mathbb{F}$ a minimal homogeneous $R-$resolution of A, and $\mathbb{K}$ the Koszul complex on a sequence of elements in R which is mapped to $\underline{\alpha}$. Since $\mathbb{F} \otimes_R \mathbb{K}$ is an $R-$resolution of of $A/(\underline{\alpha})$, one is able to observe that

$$\omega_{A/(\underline{\alpha})} \cong \left(\frac{\omega_A}{(\underline{\alpha})\omega_A}\right)\left(\sum_{i=1}^{c} \deg \alpha_i\right).$$

Huneke [11, Corollary 1.5] has proved that the ideal $I/(\underline{\alpha})$ of $A/(\underline{\alpha})$ is strongly Cohen-Macaulay. After replacing A by $A/(\underline{\alpha})$, we may assume that $c = 0$. It suffices to show that $\omega_{A/J} \cong I\omega_A$.

As before, we let $\mathbb{F}$ be a minimal homogeneous $R-$resolution of A. Let $\mathbb{G} \to A/J$ be a minimal homogeneous $R-$resolution of A/J, and $u\colon \mathbb{F} \to \mathbb{G}$ be a homogeneous morphism of complexes which lifts the natural map $\pi\colon A \to A/J$. Since I is strongly Cohen-Macaulay, we may apply [11, Proposition 1.6] in order to conclude that the ring A/J is Cohen-Macaulay. It follows that the resolutions $\mathbb{F}$ and $\mathbb{G}$ have the same length. If m denotes the common length of $\mathbb{F}$ and $\mathbb{G}$ and $\Delta = \sum_{i=1}^{t} \deg x_i$, then we see, from Convention 2.2, that u induces a homogeneous morphism

$$\pi^* = \operatorname{Ext}_R^m(\pi, R)(-\Delta)\colon\ \operatorname{Ext}_R^m(A/J, R)(-\Delta) = \omega_{A/J} \to \operatorname{Ext}_R^m(A, R)(-\Delta) = \omega_A$$

of degree zero. If $\underline{y}$ is a regular $R-$sequence of length m in the annihilator of A, then π^* may be identified with the natural map

$$\operatorname{Hom}(A/J, R/(\underline{y}))\left(\sum_{i=1}^{m} \deg y_i - \Delta\right) \hookrightarrow \operatorname{Hom}(A, R/(\underline{y}))\left(\sum_{i=1}^{m} \deg y_i - \Delta\right).$$

It follows that the map π^* is injective and the image of π^* is $0 :_{\omega_A} J$. The proof of [12, Lemma 2.1] shows that $0 :_{\omega_A} J$ **is equal to $I\omega_A$.** $\square$

Proof of Proposition 2.1. The fact that J is perfect of grade f follows from [12, Theorem 5.1]. Let $a_1, \ldots, a_f$ be a generating set for $\mathfrak{A}$ with the property that $\deg a_j = d_j$ for all j. Since each d_j is greater than the degree of every element in a minimal homogeneous generating set for I, we may deform the residual intersection $J = (\mathfrak{A} : I)$, in a homogeneous manner in a positively graded polynomial ring, in order to make $J_i = ((a_1, \ldots, a_i) : I)$ be a generic $i-$residual intersection for each i with $c \le i \le f$. (See, for example, the proof of Theorem 5.1 in [12]. The ideal J_i is a "generic" $i-$residual intersection of I if the elements $a_1, \ldots, a_i$ are linear combinations of a generating set of I where the coefficients are new variables to be adjoined to the ring containing I, cf. [12, Definition 3.1].) We prove the result by induction on $f - c$. If $f - c = 0$, then the result is well known. If $f > c$, then Huneke [11, Theorem 3.1] has proved that the ideal $(I, J_{f-1})/J_{f-1}$ of the Cohen-Macaulay ring R/J_{f-1} is strongly Cohen-Macaulay of grade one, the element a_f is regular on R/J_{f-1}, and $J_f = ((a_f, J_{f-1}) : (I, J_{f-1}))$. We may apply Lemma 2.3 in order to conclude that

$$\omega_{R/J_f} \cong \left(\frac{I\omega_{R/J_{f-1}}}{a_f\omega_{R/J_{f-1}}}\right)(d_f). \tag{2.4}$$

The induction hypothesis, together with Convention 2.2, yields a homogeneous surjection

$$\oplus_{(i)} R\left(-\left(\Delta+\sum_{k=1}^{f-c} m_{i_k}-\sum_{j=1}^{f-1} d_j\right)\right) \twoheadrightarrow \omega_{R/J_{f-1}}$$

where $\Delta=\sum_{j=1}^{t} \deg x_j$, and (i) varies over all sequences $i_1,\dots,i_{f-c}$ such that

$$1 \le i_1 \le i_2 \le \cdots \le i_{f-c} \le g.$$

It follows from (2.4) that there is a homogeneous surjection

$$\oplus_{(i)\in\mathcal{I}} R\left(-\left(\Delta+M_{(i)}-D\right)\right) \twoheadrightarrow \omega_{R/J}.$$

The above surjection is minimal; because, if $\mathfrak{m}$ is the irrelevant maximal ideal of R, then [12, Theorem 5.1] shows that

$$\left(\omega_{R/J}\right)_{\mathfrak{m}} \cong \operatorname{Sym}_{f-c+1}\left((I/\mathfrak{A})_{\mathfrak{m}}\right), \tag{2.5}$$

and it is clear that the minimal number of generators of the module on the right side of (2.5) is equal to the cardinality of $\mathcal{I}$. The proof is completed by appealing to Convention 2.2. $\square$

In the later sections we apply the Socle Lemma, together with Proposition 2.1, in order to compute the generating set for two different residual intersections. The following lemma provides the text for each argument; we apply it by "filling in the numbers".

Lemma 2.6. *Let $K \subseteq J$ be homogeneous ideals in the graded polynomial ring $R=k[x_1,\dots,x_m]$, where k is a field and each variable has degree one. Suppose that J is perfect of grade f and that the last module in the minimal homogeneous $R-$resolution of R/J has the form $F_f=\oplus_{i=1}^{r} R(-d_i)$. Suppose, further, that L is an ideal in R which satisfies*

(a) *L is generated by $m-f$ one forms;*
(b) *$R/(K+L)$ is an artinian ring; and*
(c) *the socle of $R/(K+L)$ is isomorphic, as a graded vector space, to $\oplus_{i=1}^{r} k(-(d_i-f))$;*

then $J=K$ and L is generated by a sequence which is regular on R/J.

Proof. Let $\overline{R}$ be the ring R/L, φ be the natural map from R to $\overline{R}$, and Let $\overline{}$ represent "image under φ". Since $\overline{K} \subseteq \overline{J}$, it follows that the ring $\overline{R}/\overline{J}$ is also artinian. The ring R/J is a graded Cohen-Macaulay ring of dimension equal to $m-f$. The artinian ring $\overline{R}/\overline{J}$ is obtained from R/J by modding out a sequence of $m-f$ one-forms. It follows that L is generated by a sequence which is regular on R/J. Furthermore, if $\mathbb{F}$ is the minimal homogeneous $R-$resolution of R/J, then $\mathbb{F}\otimes_R \overline{R}$ is the minimal homogeneous $\overline{R}-$resolution of $\overline{R}/\overline{J}$. Lemma 1.3 shows that the socle of $\overline{R}/\overline{J}$ is isomorphic, as a graded vector space over k, to $\oplus_{i=1}^{r} k(-(d_i-f))$. Lemma 1.1 yields that $\overline{J}=\overline{K}$; and therefore, the conclusion $J=K$ follows from Observation 2.7. $\square$

Observation 2.7. *Let $A \subseteq B$ be homogeneous ideals in the noetherian graded ring $R = \oplus_{i \ge 0} R_i$, and let $\underline{x}$ represent a sequence $x_1, \dots, x_t$ of homogeneous elements of R of positive degree. If $\underline{x}$ is a regular sequence on R/B and $B \subseteq A + (\underline{x})$, then $A = B$.*

Proof. By induction on t, we may assume that $\underline{x}$ consists of a single element x. Then, $(x) \cap B = xB$ because x is regular on R/B. Hence, $B \subseteq A + ((x) \cap B) = A + xB$, and our claim follows by Nakayama's Lemma. ☐

Section 3. A new proof that certain determinantal ideals are perfect.

In this section

(3.1) k is a field, $g, f \ge 2$ are integers, X is a $g \times (g-1)$ matrix of indeterminates, Y is $g \times f$ matrix of indeterminates, T is the generic $g \times (f+g-1)$ matrix $[X\,Y]$, R is the polynomial ring $k[X, Y]$, and $I = I_{g-1}(X)$ and $K = I_g(T)$ are determinantal ideals in R. Each variable in R is given degree one.

It has been known since at least 1960 ([**5, 15**]) that K is a perfect prime ideal in R of grade f. (Numerous other proofs of these facts may also be found in the literature; for example, [6] or [9].) We give a new proof that K is a perfect ideal in R of grade f. (One can then quickly deduce that K is a prime ideal; see, for example, the proof of Theorem 2.10 in [4].) Our proof uses the Socle Lemma to establish that K is a generic $f-$residual intersection of I; the conclusion then follows from the work of Huneke [11] and Huneke and Ulrich [12]. Huneke [11, Theorem 4.1] has already proved that K is a generic $f-$residual intersection of I; however his proof uses a fair amount of information about the ideal K. The technique of the Socle Lemma allows us to deduce that K is perfect without knowing anything about K in advance.

Theorem 3.2. *In the notation of (3.1), the $R-$ideal K is perfect of grade f.*

Proof. Let X_i represent $(-1)^{i+1}$ times the determinant of X with row i removed; let $\mathfrak{A}$ be the ideal $(A_1, \dots, A_f)$ where $[A_1, \dots, A_f] = [X_1, \dots, X_g]Y$; and let $J = (\mathfrak{A} : I)$. The ideal J is a generic $f-$residual intersection of I. The grade two perfect ideal I is known to be strongly Cohen-Macaulay ([**2, 10**]) and to satisfy the condition (G_∞). It follows from Theorem 3.3 of [12] that J is a perfect ideal in R of grade f. We prove that $J = K$. It is easy to see that $K \subseteq J$. Let $\overline{R}$ be the polynomial ring $k[x_1, \dots, x_f]$, where each variable is given degree one. We define a $k-$algebra map $\varphi\colon R \to \overline{R}$ by insisting that

$$\varphi(T) = \begin{bmatrix} x_1 & x_2 & x_3 & \dots & \dots & x_f & 0 & \dots & 0 \\ 0 & x_1 & x_2 & x_3 & \dots & \dots & x_f & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \dots & \dots & \ddots & 0 \\ 0 & \dots & 0 & x_1 & x_2 & x_3 & \dots & \dots & x_f \end{bmatrix}.$$

Observe that the kernel of φ is generated by $\dim(R) - f$ one-forms from R. Let $\overline{}$ represent "image under φ". It is well known, and easy to show (see, for example, [4, Remark 2.3]), that $\overline{K} = (x_1, \dots, x_f)^g$. Thus, $\overline{R}/\overline{K}$ is an artinian ring whose socle is isomorphic, as a graded vector space over k, to

$$k(-(g-1))^M \quad \text{where} \quad M = \binom{f+g-2}{f-1}. \tag{3.3}$$

Let F_f be the final non-zero module in the homogeneous resolution of R/J. In the notation of Proposition 2.1 we have $F_f = \oplus_{(i)\in\mathcal{I}} R(-(D-M_{(i)}))$, where $D = fg$, $M_{(i)} = (f-1)(g-1)$ for all (i), and $\mathcal{I}$ has cardinality equal to the number M of (3.3); and therefore, we conclude, from Lemma 2.6, that $J = K$. $\square$

Section 4. The residual intersection of a grade three Gorenstein ideal.

In this section

(4.1) k is a field, $g \ge 3$ is an odd integer, $f \ge 3$ is an integer, $e = f + g$, X is a $g \times g$ alternating matrix of indeterminates, Y is $g \times f$ matrix of indeterminates, and R is the polynomial ring $k[X, Y]$. Each variable in R is given degree one.

Let I be the ideal in R generated by the maximal order pfaffians of X. In other words, $I = (X_{<1>}, X_{<2>}, \dots, X_{<g>})$, where $X_{<i>}$ represents $(-1)^{i+1}$ times the pfaffian of X with row i and column i deleted. Let $\mathfrak{A}$ be the ideal $(A_{(1)}, \dots, A_{(f)})$ where

$$\left[A_{(1)}, \dots, A_{(f)}\right] = [X_{<1>}, \dots, X_{<g>}]\, Y, \tag{4.2}$$

and let $J = (\mathfrak{A} : I)$. The goal in this section is to identify a set of generators for J.

For each tuple of integers $(b) = (b_1, \dots, b_\ell)$ with

$$1 \le b_1 < b_2 < \cdots < b_\ell \le f, \quad \ell \text{ odd}, \quad \text{and } \ell \le g, \tag{4.3}$$

let $Y_{(b)}$ be the submatrix of Y consisting of columns $b_1, b_2, \dots, b_\ell$ and rows one through g, and let

$$A_{(b)} = \operatorname{Pf}\begin{bmatrix} X & Y_{(b)} \\ -\left(Y_{(b)}\right)^{\mathrm{t}} & 0 \end{bmatrix}. \tag{4.4}$$

Observe that if i is a single integer, then the meaning given to $A_{(i)}$ in (4.2) is the same as the meaning given in (4.4). Let K be the ideal in R which is generated by

$$\{A_{(b)} \mid (b) \text{ is described in (4.3)}\}.$$

In order to prove that $K \subseteq J$, we introduce a few formalisms involving pfaffians. If $Z = (z_{ij})$ is an $n \times n$ alternating matrix and c is an ordered list $c_1, \dots, c_s$ of integers with $2 \le s$ and $1 \le c_i \le n$ for all i, then let

$Z_c =$ the pfaffian of the submatrix of Z consisting of rows and columns $c_1, \dots, c_s$

in the given order. In particular, Z_c is equal to zero if s is odd, or if $c_i = c_j$ for some $i \ne j$. In this notation, the Laplace expansion for pfaffians becomes

$$\sum_{j=1}^{s} (-1)^{j+1} z_{ic_j} Z_{c_1 \dots \widehat{c_j} \dots c_s} = Z_{ic_1 \dots c_s}. \tag{4.5}$$

Lemma 4.6. *If J and K are the ideals which are defined above, then $K \subseteq J$.*

Proof. In the notation of (4.1), we let Z be the $e \times e$ alternating matrix

$$Z = \begin{bmatrix} X & Y \\ -Y^{\mathrm{t}} & 0 \end{bmatrix}.$$

Each piece of data can be viewed as a pfaffian of Z. If (b) is the tuple described in (4.3), then $A_{(b)} = Z_c$ where c is the list of integers $1, 2, \dots, g, g+b_1, g+b_2, \dots, g+b_\ell$. If i is an integer with $1 \le i \le g$, then $X_{<i>} = (-1)^{i+1} Z_c$ where c is the list of integers $1, 2, \dots, \widehat{i}, \dots, g$. We must show that $X_{<i>} A_{(b)} \in \mathfrak{A}$. Let $s = g - 1 + \ell$ and let c, equal to the list $c_1, \dots, c_s$, be $1, \dots, \widehat{i}, \dots, g, g+b_1, \dots, g+b_\ell$. Apply (4.5) twice in order to conclude

$$\sum_{j=1}^{s} (-1)^j Z_{c_j 1 \dots g} Z_{c_1 \dots \widehat{c_j} \dots c_s} = \sum_{j=1}^{s} (-1)^j \left[\sum_{k=1}^{g} (-1)^{k+1} z_{c_j k} Z_{1 \dots \widehat{k} \dots g} \right] Z_{c_1 \dots \widehat{c_j} \dots c_s}$$

$$= \sum_{k=1}^{g} (-1)^{k+1} Z_{1 \dots \widehat{k} \dots g} Z_{k c_1 \dots c_s} = Z_{1 \dots \widehat{i} \dots g} Z_{1 \dots g\, g+b_1 \dots g+b_\ell} = (-1)^{i+1} X_{<i>} A_{(b)}.$$

The proof is complete because the first expression is in $\mathfrak{A}$. □

Theorem 4.7. *If J and K are the ideals of Lemma 4.6, then $J = K$.*

Proof. The notation of (4.1) is in effect. Let Z be the $e \times e$ alternating matrix of Lemma 4.6, and let $\overline{R}$ be the polynomial ring $k[x_1, \dots, x_f]$, where each variable is given degree one. We define a $k-$algebra map $\varphi \colon R \to \overline{R}$. It suffices to define $\varphi(z_{ij})$ for all integers i and j with

(4.8) $$i + 1 \le j \le e \quad \text{and} \quad 1 \le i \le g.$$

If i and j satisfy (4.8), then define

$$\varphi(z_{ij}) = \begin{cases} x_{j-i}, & \text{if } j \le i + f \\ 0, & \text{if } i + f < j. \end{cases}$$

Observe that the kernel of φ is generated by $\dim(R) - f$ one-forms from R. Let $\overline{}$ represent "image under φ". Proposition 5.1 tells us that $\overline{R}/\overline{K}$ is an artinian ring whose socle is isomorphic, as a graded vector space over k, to

(4.9) $$k\,(-(g-1))^N \qquad \text{where} \quad N = \binom{e-3}{f-2}.$$

The ideal J is a generic $f-$residual intersection of I. It is well known, and easy to show, that the generic grade three Gorenstein ideal I satisfies the condition G_∞; Huneke [10] has shown that I is strongly Cohen-Macaulay. Theorem 3.3 of [12] guarantees that J is a perfect ideal in R of grade f. We apply Proposition 2.1 in order to calculate the back twists in the minimal homogeneous resolution of R/J. In the notation of that proposition, we have $c = 3$, $m_i = (g-1)/2$ for $1 \le i \le g$, and $d_j = (g+1)/2$ for $1 \le j \le f$. It follows that $D = f(g+1)/2$, and $M_{(i)} = (f-2)(g-1)/2$ for all $(i) \in \mathcal{I}$. We see that the difference $D - M_{(i)}$ is equal to $g + f - 1$. Furthermore, the cardinality of $\mathcal{I}$ is the integer N of (4.9). It follows that $F_f = R(-(g+f-1))^N$; and the proof is complete by Lemmas 1.3 and 2.6. □

SECTION 5. THE SOCLE OF A ZERO DIMENSIONAL SPECIALIZATION OF R/K.

In this section we prove

Proposition 5.1. *In the notation of the statement and proof of Theorem 4.7, $\overline{R}/\overline{K}$ is an artinian ring whose socle is isomorphic to the graded vector space given in (4.9).*

Our proof of Proposition 5.1 is based on the following two calculations. Observation 5.2 is a combinatorial fact. We consider the binomial coefficient $\binom{a}{b}$ to be meaningful for all integers $a \ge 0$ and b. If $a < b$, or if $b < 0$, then $\binom{a}{b} = 0$. Lemma 5.4 is where the serious work in this argument takes place.

Observation 5.2. *If ε, s, and m are integers with $\varepsilon = 0$ or 1, $1 \le s$ and $0 \le m$, then*

$$\sum_{r=1}^{m+1} \binom{s+m-r}{s-1}\binom{s}{2r-1-\varepsilon} = \binom{s+2m-\varepsilon}{s-1}. \tag{5.3}$$

Lemma 5.4. *Adopt the notation of Theorem 4.7. Let $n = \frac{g-1}{2}$. If $G_2 \to G_1 \to \overline{K} \to 0$ is the minimal homogeneous $\overline{R}-$presentation of $\overline{K}$, then*

$$G_1 = \sum_{r=1}^{n+1} \overline{R}(-(n+r))^{e(r)} \quad \textit{where} \quad e(r) = \binom{f}{2r-1}, \quad \textit{and}$$

G_2 has the form $\oplus_i \overline{R}(-d_i)$ where $d_i \ge 2n+2$ for all i.

If we assume 5.2 and 5.4 for the time being, then Proposition 5.1 follows quickly.

Proof of Proposition 5.1. Use Lemma 5.4 and Observation 5.2 in order to see that

$$\dim(\overline{R}/\overline{K})_{2n+1} = \binom{2n+f}{f-1} - \sum_{r=1}^{n+1}\binom{n-r+f}{f-1}\binom{f}{2r-1} = 0. \tag{5.5}$$

The graded polynomial ring $\overline{R}$ is generated as a $k-$algebra by $\overline{R}_1$; consequently $(\overline{R}/\overline{K})_\ell = 0$ for all $\ell \ge 2n+1$. At this point, we conclude that $\overline{R}/\overline{K}$ is an artinian ring, and that $(\overline{R}/\overline{K})_{2n}$ is contained in $\operatorname{Socle}(\overline{R}/\overline{K})$. A closer examination of Lemma 5.4 yields that

$$(\overline{R}/\overline{K})_{2n} = \operatorname{Socle}(\overline{R}/\overline{K}). \tag{5.6}$$

Indeed, the $\overline{R}-$ideal $\overline{K}$ is perfect of grade f. If $\mathbb{G}$ is the minimal homogeneous $\overline{R}-$resolution of $\overline{R}/\overline{K}$, then we conclude, using Lemma 5.4, that G_f has the form $\oplus_i \overline{R}(-d_i)$ where every $d_i \ge 2n+f$. The assertion of line (5.6) now follows from Lemma 1.3. We complete the proof by observing that the technique of (5.5) shows that $\dim(\overline{R}/\overline{K})_{2n}$ is the integer N of (4.9). $\square$

In the course of establishing Observation 5.2 we use the following well known identity:

$$\sum_{\ell=0}^{m}\binom{s-1+\ell}{s-1}=\binom{s+m}{s}. \tag{5.7}$$

Proof of Observation 5.2. Let $\Psi(s,m,\varepsilon)$ represent the left side of (5.3). We prove the result by induction on s. Both sides of (5.3) are equal to 1 if $s=1$. Some manipulations involving binomial coefficients are necessary before we can continue this induction. Reverse the order of summation and use (5.7) in order to see that

$$\sum_{k=0}^{m-1}\Psi(s,k,\varepsilon)=\sum_{r=1}^{m}\binom{s+m-r}{s}\binom{s}{2r-1-\varepsilon}. \tag{5.8}$$

Observe further that

$$\Psi(s+1,m,\varepsilon)=\sum_{k=0}^{m}\Psi(s,k,1)\ +\sum_{k=0}^{m-\varepsilon}\Psi(s,k,0). \tag{5.9}$$

Indeed, $\Psi(s+1,m,\varepsilon)$ is equal to $\Psi_1+\Psi_2$ where

$$\Psi_i=\sum_{r=1}^{m+1}\binom{s+1+m-r}{s}\binom{s}{2r-i-\varepsilon}.$$

Line (5.8) gives $\Psi_1=\sum_{k=0}^{m}\Psi(s,k,\varepsilon)$ and $\Psi_2=\sum_{k=0}^{m-\varepsilon}\Psi(s,k,1-\varepsilon)$.

We now complete our proof of (5.3). Assume, by induction, that (5.3) holds for a fixed value of s and all values of m and ε. We may apply (5.9), the induction hypothesis, and (5.7), in order to see that $\Psi(s+1,m,\varepsilon)$ is equal to

$$\sum_{k=0}^{m}\binom{s+2k-1}{s-1}+\sum_{k=0}^{m-\varepsilon}\binom{s+2k}{s-1}=\sum_{\ell=0}^{2m+1-\varepsilon}\binom{s-1+\ell}{s-1}=\binom{s+2m+1-\varepsilon}{s}. \qquad \square$$

We prove Lemma 5.4 by showing that there are no relations in $\overline{R}$, of degree less than or equal to g, on the set of generators $\{\overline{A_{(b)}}\mid (b)\text{ is described in (4.3)}\}$ for $\overline{K}$. This statement is established in Proposition 5.11, where relations on a more general collection of pfaffians are considered. We must introduce some notation in order to state Proposition 5.11. For each positive integer s, let $S^{(s)}$ be the polynomial ring $k[x_1,\dots,x_s]$, where each variable is given degree one. (In particular, $S^{(f)}=\overline{R}$.) The odd integer $g\ge 3$ remains fixed. For each positive integer s, we define a $(g+s)\times(g+s)$ alternating matrix $Z^{(s)}$ with entries from $S^{(s)}$. The matrix $Z^{(s)}$ is the difference $M-M^{\mathrm{t}}$, where $M=(m_{ij})$ is defined by

$$m_{ij}=\begin{cases} x_{j-i}, & \text{if } i+1\le j\le i+s \quad\text{and}\quad 1\le i\le g\\ 0, & \text{otherwise.}\end{cases}$$

For example, if $g=3$, then

$$Z^{(4)}=\left[\begin{array}{ccc|cccc} 0 & x_1 & x_2 & x_3 & x_4 & 0 & 0\\ -x_1 & 0 & x_1 & x_2 & x_3 & x_4 & 0\\ -x_2 & -x_1 & 0 & x_1 & x_2 & x_3 & x_4\\ \hline -x_3 & -x_2 & -x_1 & 0 & 0 & 0 & 0\\ -x_4 & -x_3 & -x_2 & 0 & 0 & 0 & 0\\ 0 & -x_4 & -x_3 & 0 & 0 & 0 & 0\\ 0 & 0 & -x_4 & 0 & 0 & 0 & 0 \end{array}\right].$$

In particular, $Z^{(f)}$ is equal to $\overline{Z}$. If c is an ordered list $c_1,\dots,c_r$ of integers, then $Z_c^{(s)}$ is the pfaffian of a submatrix of $Z^{(s)}$ as described above (4.5). If m is a positive integer, then $[m]$ refers to the ordered list $1,2,\dots,m$.

Convention 5.10. *Let m and s be positive integers with $1\le m\le g$, and let c represent the list of indices $c_1,\dots,c_r$. If all of the following conditions hold:*

(a) *$m+r$ is even,*
(b) *at least half of the indices $1,\dots,m,c_1,\dots,c_r$ are less than or equal to g, and*
(c) *$m+1\le c_1<c_2<\cdots<c_r\le m+s$,*

then we write $c\in\mathfrak{S}(m,s)$.

Proposition 5.11. *If m and s are positive integers with $1\le m\le g$, then there are no relations in $S^{(s)}$ of degree less than or equal to m of the form*

$$\sum_c a_c Z_{[m]c}^{(s)}=0,$$

where c varies over all lists of indices in $\mathfrak{S}(m,s)$.

Before proving the proposition, we make sure that its meaning and significance are clear. The listed equation represents a "relation of degree d" if each product $a_cZ_{[m]c}^{(s)}$ is a homogeneous element of $S^{(s)}$ of degree d. Moreover, we see that Proposition 5.11 implies Lemma 5.4. Indeed, if $m=g$ and $s=f$, then

$$\{Z_{[g]c}^{(f)}\mid c\in\mathfrak{S}(g,f)\}=\{\overline{A_{(b)}}\mid (b)\text{ is described in (4.3)}\}.$$

In the course of proving Proposition 5.11 it is convenient to partition the set of lists $\mathfrak{S}(m,s)$ into two disjoint subsets. Suppose that c represents the list $c_1,\dots,c_r$ from $\mathfrak{S}(m,s)$. We write

$$c\in\mathfrak{S}_e(m,s),\text{if } c_1=m+1,\quad\text{and}\quad c\in\mathfrak{S}_n(m,s),\text{if } c_1\ne m+1.$$

(In the second case, c_1 is, in fact, greater than $m+1$.)

Proof of Proposition 5.11. The proof proceeds by induction on m and s. The result is obvious when $m=1$ and s is arbitrary; and also when $s=1$ and m is arbitrary.

We assume, by induction, that the proposition holds for $(m, s-1)$ and $(m-1, s)$, where m and s are fixed integers with $1 < m \le g$ and $1 < s$. Suppose that

$$\sum_c a_c Z^{(s)}_{[m]c} = 0, \tag{5.12}$$

is a relation in $S^{(s)}$ of degree less than or equal to m, where c varies over all lists of indices in $\mathfrak{S}(m, s)$. We will prove that each $a_c = 0$. We begin by partitioning the set of lists $\mathfrak{S}(m, s)$ into two disjoint subsets:

$$\mathfrak{S}(m, s) = \mathfrak{S}(m, s-1) \cup \{c',\, m+s \mid c' \in \mathfrak{S}_n(m-1, s)\}.$$

Indeed, let c be the list $c_1, \dots, c_r$ from $\mathfrak{S}(m, s)$. If $c_r \ne m+s$, then $c \in \mathfrak{S}(m, s-1)$. If $c_r = m+s$ and c' represents $c_1, \dots, c_{r-1}$, then $c' \in \mathfrak{S}_n(m-1, s)$. We use our latest partition of $\mathfrak{S}(m, s)$ in order to rewrite (5.12) as

$$\sum_c a_c Z^{(s)}_{[m]c} \;+\; \sum_{c'} a_{c'\,m+s} Z^{(s)}_{[m]c'\,m+s} = 0, \tag{5.13}$$

where $c \in \mathfrak{S}(m, s-1)$ and $c' \in \mathfrak{S}_n(m-1, s)$. It suffices to prove that $a_c = 0$ and $a_{c'\,m+s} = 0$ for all c and c'. Use (4.5) in order to expand $Z^{(s)}_{[m]c'\,m+s}$ down the last column:

$$Z^{(s)}_{[m]c'\,m+s} = \pm\, x_s Z^{(s)}_{[m-1]c'} + \sum_{c''} h_{c''} Z^{(s)}_{[m]c''} \tag{5.14}$$

for some polynomials $h_{c''} \in S^{(s)}$ where c'' varies over all lists of indices in $\mathfrak{S}(m, s-1)$. When the equation of line (5.14) is substituted into (5.13) we obtain the relation:

$$\sum_c a'_c Z^{(s)}_{[m]c} + \sum_{c'} a'_{c'\,m+s}\, x_s Z^{(s)}_{[m-1]c'} = 0 \tag{5.15}$$

where $c \in \mathfrak{S}(m, s-1)$ and $c' \in \mathfrak{S}_n(m-1, s)$. The coefficient $a'_{c'\,m+s}$ is equal to $\pm\, a_{c'\,m+s}$. The coefficient a'_c differs from a_c by some element from the ideal

$$(\{a_{c'\,m+s} \mid c' \in \mathfrak{S}_n(m-1, s)\})\,.$$

Consequently, it suffices to show that $a'_c = 0$ and $a'_{c'\,m+s} = 0$ for all c and c'.

Consider the $S^{(s-1)}$–algebra homomorphism $\varphi\colon S^{(s)} \to S^{(s-1)}$ which sends x_s to zero. Observe that φ carries $Z^{(s)}_c$ to $Z^{(s-1)}_c$ if $g+s$ does not appear in the list c, and φ carries $Z^{(s)}_c$ to 0 if $g+s$ does appear in the list c. When φ is applied to the equation of (5.15), we obtain a relation of degree less than or equal to m in $S^{(s-1)}$. The induction hypothesis, applied to the pair $(m, s-1)$, yields that a'_c is divisible by x_s for all c in $\mathfrak{S}(m, s-1)$. Write $a'_c = x_s a''_c$. The element x_s is regular in the domain $S^{(s)}$, so we may divide the relation of (5.15) by x_s in order to obtain the relation

$$\sum_c a''_c Z^{(s)}_{[m]c} + \sum_{c'} a'_{c'\,m+s} Z^{(s)}_{[m-1]c'} = 0 \tag{5.16}$$

in $S^{(s)}$ of degree $m-1$ or less, where c varies over $\mathfrak{S}(m, s-1)$ and c' varies over $\mathfrak{S}_n(m-1, s)$. Once again, the proof is finished when we show that all a''_c and all $a'_{c'\,m+s}$ are zero. Observe that

$$\text{the list } c \text{ is in } \mathfrak{S}(m, s-1) \iff \text{the list } m\,c \text{ is in } \mathfrak{S}_e(m-1, s).$$

If $c \in \mathfrak{S}(m, s-1)$, then let d represent the list $m\,c$ and b_d represent the polynomial a''_c. The first summand in (5.16) is

$$\sum_d b_d Z^{(s)}_{[m-1]d} \tag{5.17}$$

where d varies over $\mathfrak{S}_e(m-1, s)$. Similarly, the second summand of (5.16) is given in (5.17), where, this time, d varies over $\mathfrak{S}_n(m-1, s)$. The set of lists $\mathfrak{S}(m-1, s)$ is the disjoint union of $\mathfrak{S}_e(m-1, s)$ and $\mathfrak{S}_n(m-1, s)$. The induction hypothesis, applied to the pair $(m-1, s)$ yields that $b_d = 0$ for all $d \in \mathfrak{S}(m-1, s)$; and therefore, the proof is complete. □

For a particular ring it is often useful to know an explicit system of parameters. In the course of our proof, we have identified one for the ring R/K of Section 4. Furthermore, we have learned much qualitative information about the $R-$resolution of R/K. In particular, the resolution is linear from position two until the end. (The entire resolution of R/K may be found in [14].)

Corollary 5.18. *Adopt the notation of (4.1). If K is the ideal defined below (4.4), then the following statements hold.*

(a) *The $R-$ideal K is perfect of grade f.*
(b) *The ring $\overline{R}/\overline{K}$ of the proof of Theorem 4.7 is a zero-dimensional specialization of the ring R/K.*
(c) *Let $n = (g-1)/2$, and N be the integer defined in (4.9). If $\mathbb{G}$ is the minimal homogeneous $R-$resolution of R/K, then G_1 is described in Lemma 5.4, G_f is equal to $R(-(2n+f))^N$, and if $2 \le i \le f-1$, then $G_i = R((-2n+i))^{p_i}$ for some numbers p_i.*

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