THE POINCARÉ SERIES OF EVERY FINITELY GENERATED MODULE OVER A CODIMENSION FOUR ALMOST COMPLETE INTERSECTION IS A RATIONAL FUNCTION

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Abstract. Let $(R,\mathfrak{M},k)$ be a regular local ring in which two is a unit and let $A=R/J$, where J is a five generated grade four perfect ideal in R. We prove that the Poincaré series $P_A^M(z)=\sum_{i=0}^{\infty}\dim_k \operatorname{Tor}_i^A(M,k)z^i$ is a rational function for all finitely generated $A-$modules M. We also prove that the Eisenbud conjecture holds for A, that is, if M is an $A-$module whose Betti numbers are bounded, then the minimal resolution of M by free $A-$modules is eventually periodic of period at most two.

Let A be a quotient of a regular local ring $(R,\mathfrak{M},k)$. If any of the following conditions hold:

(a) $\operatorname{codim} A \le 3$, or
(b) $\operatorname{codim} A = 4$ and A is Gorenstein, or
(c) A is one link from a complete intersection, or
(d) A is two links from a complete intersection and A is Gorenstein,

then it has been shown in [4] and [8] that all of the following conclusions hold:

(1) The Poincaré series $P_A^M(z)=\sum_{i=0}^{\infty}\dim_k \operatorname{Tor}_i^A(k,M)\,z^i$ is a rational function for all finitely generated $A-$modules M.
(2) If R contains the field of rational numbers, then the Herzog Conjecture [14] holds for the ring A. That is, the cotangent modules $T_i(A/R,A)$ vanish for all large i if and only if A is a complete intersection.
(3) The Eisenbud Conjecture [10] holds for the ring A. That is, if M is a finitely generated $A-$module whose Betti numbers are bounded, then the minimal resolution of M eventually becomes periodic of period at most two.

In the present paper we prove that conclusions (1) – (3) all hold in the presence of hypothesis

(e) A is an almost complete intersection of codimension four in which two is a unit.

In each of the cases (a) – (e), there are three steps to the process:

(i) one proves that the minimal $R-$resolution of A is a DG$-$algebra;
(ii) one classifies the Tor$-$algebras $\operatorname{Tor}_\bullet^R(A,k)$; and

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(iii) one completes the proof of (1) – (3).

For hypothesis (e), step (i) was begun in [19] and [20], and was completed in [18]; step (ii) was carried out in [17]; and step (iii) is contained in the present paper.

In the following section by section description of the paper, let $(R, \mathfrak{M}, k)$ be a regular local ring and $(A, \mathfrak{m}, k)$ be the quotient R/J, where J is a grade four almost complete intersection ideal in R. Section 1 is a review of the classification of the Tor−algebras $\operatorname{Tor}^R_\bullet(A, k)$. Many of the results in this paper are obtained by proving that the appropriate DGΓ−algebra is Golod. The definition and properties of Golod algebras may be found in section 2. We compute the Poincaré series $P^k_A(z)$ in section 3. In section 4, we apply a new result (Theorem 4.1), due to Avramov, in order to prove that the Poincaré series $P^M_A(z)$ is rational for all finitely generated A−modules M. The growth of the Betti numbers of M is investigated in section 5. The proof, in [8], that property (1) holds in the presence of any of conditions (a) – (d), depends on proving that there is a Golod homomorphism $C \twoheadrightarrow A$ from a complete intersection C onto A. In section 6 we observe that while the technique of [8] applies to most codimension four almost complete intersections A, there do exist A for which it does not apply. It follows that the generalization in Theorem 4.1 of the technique from [8] is essential to the completion of this paper.

In this paper "ring" means commutative noetherian ring with one. The *grade* of a proper ideal I in a ring R is the length of the longest regular sequence on R in I. The ideal I of R is called *perfect* if the grade of I is equal to the projective dimension of the R−module R/I. A grade g ideal I is called a *complete intersection* if it can be generated by g generators. Complete intersection ideals are necessarily perfect. The grade g ideal I is called an *almost complete intersection* if it is a **perfect** ideal which is **not** a complete intersection and which can be generated by $g + 1$ generators. We use the concepts "graded k−algebra", "trivial module", and "trivial extension" in the usual manner; see [17]. If $S_\bullet$ is a divided power algebra, then $S_\bullet{<}x{>}$ represents a divided power extension of $S_\bullet$. The algebra $(S_\bullet = \bigoplus_{i\ge 0} S_i,\, d)$ is a DGΓ−algebra if

(a) the multiplication $S_i \times S_j \to S_{i+j}$ is graded−commutative ($s_i s_j = (-1)^{ij} s_j s_i$ for $s_k \in S_k$ and $s_i s_i = 0$ for i odd) and associative,
(b) the differential $d\colon S_i \to S_{i-1}$ satisfies $d(s_i s_j) = d(s_i)s_j + (-1)^i s_i d(s_j)$,
(c) for each homogeneous element s in $S_\bullet$ of positive even degree, there is an associated sequence of elements $s^{(0)}, s^{(1)}, s^{(2)}, \dots$ which satisfies $s^{(0)} = 1$, $s^{(1)} = s$, $\deg s^{(k)} = k \deg s$, as well as a list of other axioms (see [13, Definition 1.7.1]), and
(d) $d(s^{(k)}) = (ds)s^{(k-1)}$ for each homogeneous $s \in S_\bullet$ of positive even degree.

Section 1. The classification of the Tor−algebras.

If k is any field, then let $\mathbf{A} - \mathbf{F}^\star$ be the DGΓ−algebras over k which are defined in Table 1. Further numerical information about (and alternate descriptions of) these algebras may be found in [17]. (Table 1 and [17] define the same algebras $S_\bullet = \mathbf{A}, \dots, \mathbf{F}^\star$ in all cases, except when $\operatorname{char} k = 2$ and $S_\bullet = \mathbf{F}^\star$. All of the results in [17] and almost all of the results in the present paper assume $\operatorname{char} k \neq 2$; consequently, one may use either definition of $\mathbf{F}^\star$ in these places. However, the correct definition of $\mathbf{F}^\star$ is given in Table 1; see Example 3.8.)

The following result is an extension of the main result in [17]. The new information is the observation that all of the Betti numbers of the R−module R/J are determined by the form of $S_\bullet$ together with the Cohen-Macaulay type of R/J.

Theorem 1.1. *Let $(R, \mathfrak{M}, k)$ be a local ring in which 2 is a unit. Assume that every element of k has a square root in k. Let J be a grade four almost complete intersection ideal in R, and let $T_\bullet$ be the graded k−algebra $\operatorname{Tor}^R_\bullet(R/J, k)$. Then there is a parameter p, q, or r which satisfies*

$$0 \le p, \quad 2 \le q \le 3, \quad \textit{and} \quad 2 \le r \le 5, \tag{1.2}$$

an algebra $S_\bullet$ from the list

$$\mathbf{A},\ \mathbf{B}[p],\ \mathbf{C}[p],\ \mathbf{C}^{(2)},\ \mathbf{C}^\bigstar,\ \mathbf{D}[p],\ \mathbf{D}^{(2)},\ \mathbf{E}[p],\ \mathbf{E}^{(q)},\ \mathbf{F}[p],\ \mathbf{F}^{(r)},\ \mathbf{F}^\bigstar,$$

and a positively graded vector space W such that, $T_\bullet$ is isomorphic (as a graded k−algebra) to the trivial extension $S_\bullet \ltimes W$ of $S_\bullet$ by the trivial $S_\bullet$−module W. Furthermore, W is completely determined by $\dim_k T_4$ together with the subalgebra $k[T_1]$ of $T_\bullet$. In particular, if $\dim_k T_4 = t$, then $\dim_k T_3 = \dim_k T_2 + t - 4$, where $\dim_k T_2$ is given in the following table.

(1.3)

$k[T_1]$	$\dim_k T_2$
$\mathbf{A} \ltimes k(-1)$	$t+6$
$\mathbf{B}[0]$	$t+7$
$\mathbf{C}[0]$	$t+7$
$\mathbf{D}[0]$	$t+8$
$\mathbf{E}[0]$	$t+9$
$\mathbf{F}[0]$	$t+10$

Remark. The classification of $k[T_1]$ and the chart which relates $\dim T_2$ and $\dim T_4$ both remain valid, even if k is not closed under the square root operation.

Proof. In light of [17], it suffices to verify the table which gives $\dim_k T_2$ in terms of t. Let S be any four dimensional subspace of T_1. Lemma 3.9 of [18] uses a linkage argument to produce vector spaces $\overline{L}_1$ and $\overline{L}_3$, and a linear transformation $\overline{\beta}_3 \colon \overline{L}_3 \to k^4$ such that

(a) $T_2 = S^2 \oplus \overline{L}_1$,
(b) $T_4 = \ker \overline{\beta}_3$,
(c) $\dim_k \overline{L}_1 = \dim_k \overline{L}_3$, and
(d) $\dim_k S^3 = 4 - \operatorname{rank} \overline{\beta}_3$.

A quick calculation yields

$$\dim_k T_2 = \dim_k T_4 + \dim_k S^2 - \dim_k S^3 + 4.$$

Let S be the subspace (x_1, x_2, x_3, x_4) of T_1. The following table completes the proof.

$k[T_1]$	$\dim_k S^2$	$\dim_k S^3$
$\mathbf{A} \ltimes k(-1)$	6	4
$\mathbf{B}[0]$	6	3
$\mathbf{C}[0]$	5	2
$\mathbf{D}[0]$	6	2
$\mathbf{E}[0]$	6	1
$\mathbf{F}[0]$	6	0

$\square$

We conclude this section by identifying the Tor$-$algebras from Table 1 which correspond to hypersurface sections. The proof of Proposition 1.4 follows the proof of (3.3) and (3.4) in [4]; it does not use the classification from [17]. On the other hand, the proof of Observation 1.5 does use [17]; the chief significance of this result is that it shows that if the Tor$-$algebra $T_\bullet$ has the form $\mathbf{C}^\bigstar \ltimes W$, then W must be zero.

Proposition 1.4. *Let J be a grade four almost complete intersection ideal in the local ring $(R, \mathfrak{M}, k)$. Let $T_\bullet = \operatorname{Tor}_\bullet^R(R/J, k)$ and $t = \dim_k T_4$. The following statements are equivalent.*

(a) *The ideal J is a hypersurface section; that is, there exists an ideal $J' \subseteq R$ and an element $a \in R$, such that a is regular on R/J' and $J = (J', a)$.*
(b) *There is a nonzero element h in T_1 such that $T_\bullet$ is a free module over the subalgebra $k\langle h\rangle$.*
(c) *The algebra $T_\bullet$ is isomorphic to $\mathbf{B}[t]$, $\mathbf{C}[t]$, or $\mathbf{C}^\bigstar$.*

Proof. (a) $\implies$ (c) The element a is regular on R; consequently, J' is a grade three almost complete intersection. Such ideals have been classified by Buchsbaum and Eisenbud [9, Proposition 5.3]. The computation of $T'_\bullet = \operatorname{Tor}_\bullet^R(R/J', k)$ and $T_\bullet = T'_\bullet \otimes_k \operatorname{Tor}_\bullet^R(R/(a), k)$ follows quickly. Indeed, it is clear that t is equal to $\dim_k T'_3$; thus, in the language of [8, Theorem 2.1], we have

$$T'_\bullet = \begin{cases} \mathbf{H}(3,2), & \text{if } t = 2, \\ \mathbf{TE}, & \text{if } t \geq 3 \text{ is odd, and} \\ \mathbf{H}(3,0) & \text{if } t \geq 4 \text{ is even,} \end{cases} \quad \text{and} \quad T_\bullet = \begin{cases} \mathbf{C}^\bigstar, & \text{if } t = 2, \\ \mathbf{B}[t], & \text{if } t \geq 3 \text{ is odd, and} \\ \mathbf{C}[t], & \text{if } t \geq 4 \text{ is even.} \end{cases}$$

(c) $\implies$ (b) It is obvious that each of the three listed algebras is a free module over the subalgebra $k\langle x_1\rangle$.

(b) $\implies$ (a) Let ψ represent the composition $J \to J/\mathfrak{m}J \xrightarrow{\cong} T_1$, and select an element $a \in J$ such that a is a regular element of R and $\psi(a) = h$. Avramov [4] has proved that $J/(a)$ is a grade three almost complete intersection ideal in $R/(a)$. The structure theorem of Buchsbaum and Eisenbud [9] produces the required grade three almost complete intersection J' in R. $\square$

Observation 1.5. *If the notation and hypotheses of Theorem 1.1 are adopted, then the following statements are equivalent.*

(a) *The algebra $T_\bullet$ is isomorphic to $\mathbf{C}^\bigstar \ltimes W$ for some trivial $\mathbf{C}^\bigstar-$module W.*

The definition of the algebras $\mathbf{A} - \mathbf{F}^\star$

Each $k-$algebra $S_\bullet = \bigoplus_{i=0}^4 S_i$ is a DG$\Gamma-$algebra with $S_0 = k$ and $d_i = \dim_k S_i$. Select bases $\{x_i\}$ for S_1, $\{y_i\}$ for S_2, $\{z_i\}$ for S_3, and $\{w_i\}$ for S_4. View S_2 as the direct sum $S_2' \oplus S_1^2$. Every product of basis vectors which is not listed has been set equal to zero. The parameters p, q, and r satisfy (1.2). The differential in $S_\bullet$ is identically zero.

$S_\bullet$	d_1	d_2	d_3	d_4	$S_1 \times S_1$	$S_1 \times S_1 \times S_1$	$S_1 \times S_2'$	$S_1 \times S_3$	$S_2^{(2)}$
$\mathbf{A}$	4	6	4	0	(a)	(a$'$)	0	0	0
$\mathbf{B}[p]$	5	$p+7$	$2p+3$	p	(b) with $\ell = p$	(b$'$) with $\ell = 2p$	(g)	(g$'$)	0
$\mathbf{C}[p]$	5	$p+7$	$2p+3$	p	(c) with $\ell = p$	(c$'$) with $\ell = 2p$	(g)	(g$'$)	0
$\mathbf{C}^{(2)}$	5	8	7	1	(c) with $\ell = 1$	(c$'$) with $\ell = 4$	(h) with $j = 2$	(h$'$) with $j = 2$	0
$\mathbf{C}^\star$	5	9	7	2	(c) with $\ell = 2$	(c$'$) with $\ell = 4$	(i)	(i$'$)	(i$'$)
$\mathbf{D}[p]$	5	$p+8$	$2p+2$	p	(d) with $\ell = p$	(d$'$) with $\ell = 2p$	(g)	(g$'$)	0
$\mathbf{D}^{(2)}$	5	9	6	1	(d) with $\ell = 1$	(d$'$) with $\ell = 4$	(h) with $j = 2$	(h$'$) with $j = 2$	0
$\mathbf{E}[p]$	5	$p+9$	$2p+1$	p	(e) with $\ell = p$	(e$'$) with $\ell = 2p$	(g)	(g$'$)	0
$\mathbf{E}^{(q)}$	5	10	$2q+1$	1	(e) with $\ell = 1$	(e$'$) with $\ell = 2q$	(h) with $j = q$	(h$'$) with $j = q$	0
$\mathbf{F}[p]$	5	$p+10$	$2p$	p	(f) with $\ell = p$	0	(g)	(g$'$)	0
$\mathbf{F}^{(r)}$	5	11	$2r$	1	(f) with $\ell = 1$	0	(h) with $j = r$	(h$'$) with $j = r$	0
$\mathbf{F}^\star$	5	12	10	2	(f) with $\ell = 2$	0	(h) with $j = 5$	(h$'$) with $j = 5$	(j)

KEY:

(a) $x_1x_2 = y_1,\ x_1x_3 = y_2,\ x_1x_4 = y_3,\ x_2x_3 = y_4,\ x_2x_4 = y_5,\ x_3x_4 = y_6$

(a$'$) $x_1x_2x_3 = z_1,\ x_1x_2x_4 = z_2,\ x_1x_3x_4 = z_3,\ x_2x_3x_4 = z_4$

(b) $x_1x_2 = y_{\ell+1},\ x_1x_3 = y_{\ell+2},\ x_1x_4 = y_{\ell+3},\ x_1x_5 = y_{\ell+4},\ x_2x_3 = y_{\ell+5},\ x_2x_4 = y_{\ell+6},\ x_3x_4 = y_{\ell+7}$

(b$'$) $x_1x_2x_3 = z_{\ell+1},\ x_1x_2x_4 = z_{\ell+2},\ x_1x_3x_4 = z_{\ell+3}$

(c) $x_1x_2 = y_{\ell+1},\ x_1x_3 = y_{\ell+2},\ x_1x_4 = y_{\ell+3},\ x_1x_5 = y_{\ell+4},\ x_2x_3 = y_{\ell+5},\ x_2x_4 = y_{\ell+6},\ x_2x_5 = y_{\ell+7}$

(c$'$) $x_1x_2x_3 = z_{\ell+1},\ x_1x_2x_4 = z_{\ell+2},\ x_1x_2x_5 = z_{\ell+3}$

(d) $x_1x_2 = y_{\ell+1},\ x_1x_3 = y_{\ell+2},\ x_1x_4 = y_{\ell+3},\ x_1x_5 = y_{\ell+4},\ x_2x_3 = y_{\ell+5},\ x_2x_4 = y_{\ell+6},\ x_2x_5 = y_{\ell+7},\ x_3x_4 = y_{\ell+8}$

(d$'$) $x_1x_2x_3 = z_{\ell+1},\ x_1x_2x_4 = z_{\ell+2}$

(e) $x_1x_2 = y_{\ell+1},\ x_1x_3 = y_{\ell+2},\ x_1x_4 = y_{\ell+3},\ x_1x_5 = y_{\ell+4},\ x_2x_3 = y_{\ell+5},\ x_2x_4 = y_{\ell+6},\ x_2x_5 = y_{\ell+7},\ x_3x_4 = y_{\ell+8},\ x_3x_5 = y_{\ell+9}$

(e$'$) $x_1x_2x_3 = z_{\ell+1}$,

(f) $x_1x_2 = y_{\ell+1},\ x_1x_3 = y_{\ell+2},\ x_1x_4 = y_{\ell+3},\ x_1x_5 = y_{\ell+4},\ x_2x_3 = y_{\ell+5},\ x_2x_4 = y_{\ell+6},\ x_2x_5 = y_{\ell+7},\ x_3x_4 = y_{\ell+8},\ x_3x_5 = y_{\ell+9},\ x_4x_5 = y_{\ell+10}$

(g) $x_1y_i = z_i$ for $1 \le i \le p$,

(g$'$) $x_1z_{p+i} = w_i$ for $1 \le i \le p$,

(h) $x_iy_1 = z_i$ for $1 \le i \le j$,

(h$'$) $x_iz_{j+i} = w_1$ for $1 \le i \le j$,

(i) $x_1y_1 = z_1,\ x_1y_2 = z_2,\ x_2y_1 = z_3,\ x_2y_2 = z_4$

(i$'$) $x_1x_2y_1 = w_1,\ x_1x_2y_2 = w_2$,

(j) $y_1y_2 = w_1,\ y_1^{(2)} = w_2$.

Table 1

(b) *There exist elements $a_1, a_2 \in R$ and a three generated, grade two perfect* *ideal* $J' \subseteq R$ *such that* a_1, a_2 *is a regular sequence on* R/J' *and* $J = (J', a_1, a_2)$.

Furthermore, if the above conditions hold, then $W = 0$.

Proof. A straightforward calculation shows that if condition (b) holds, then $T_\bullet = \mathbf{C}^\bigstar$. On the other hand, if statement (a) holds, then Table 4.13 and Lemma 4.14 in [17] show that cases two and three are not relevant; and hence, case one applies. It follows that J is equal to $K : I$ for complete intersection ideals K and I with

$$\dim_k \left(\frac{K + \mathfrak{M}I}{\mathfrak{M}I} \right) = 2.$$

It is not difficult to show (see, for example, [8, Section 3]) that J has the form of (b). $\square$

Section 2. Golod DGΓ−algebras.

Many of the theorems in sections 3 and 5 are proved by showing that certain DGΓ−algebras are Golod. In the present section we collect the necessary definitions and facts about Golod algebras; most of this information may be found in [3] or [8].

Notation 2.1. Let $(\boldsymbol{P} = \bigoplus_{i \geq 0} \boldsymbol{P}_i, d)$ be a DGΓ−algebra with $(\boldsymbol{P}_0, \mathfrak{m}, k)$ a local ring and $H_0(\boldsymbol{P}) = k$. Assume that $\boldsymbol{P}_i$ is a finitely generated $\boldsymbol{P}_0$−module for all i. Let $Z(\boldsymbol{P})$, $B(\boldsymbol{P})$, and $H(\boldsymbol{P})$ represent the cycles, boundaries, and homology of $\boldsymbol{P}$, respectively. Let

$$\varepsilon \colon \boldsymbol{P} \to \frac{\boldsymbol{P}}{\mathfrak{m} \oplus (\oplus_{i \geq 1} \boldsymbol{P}_i)} = k$$

be a fixed augmentation. The complex map ε (where k is viewed as a complex concentrated in degree zero) induces augmentations $\varepsilon \colon H(\boldsymbol{P}) \to H(k) = k$ and $\varepsilon \colon Z(\boldsymbol{P}) \to Z(k) = k$. Let $I\_\_$ represent the kernel of the augmentation map; in particular, $I\boldsymbol{P} = \mathfrak{m} \oplus (\oplus_{i \geq 1} \boldsymbol{P}_i)$, $IH(\boldsymbol{P}) = \oplus_{i \geq 1} H_i(\boldsymbol{P})$, and $IZ(\boldsymbol{P}) = \mathfrak{m} \oplus (\oplus_{i \geq 1} Z_i(\boldsymbol{P}))$.

Definition 2.2. Adopt the notation of (2.1). A (possibly infinite) subset $\mathcal{S}$ of homogeneous elements of $IH(\boldsymbol{P})$ is said to admit a *trivial Massey operation* if there exists a function γ defined on the set of finite sequences of elements of $\mathcal{S}$ (with repetitions) taking values in $I\boldsymbol{P}$, subject to the following conditions.

(1) If h is in $\mathcal{S}$, then $\gamma(h)$ is a cycle in $Z(\boldsymbol{P})$ which represents h in $H(\boldsymbol{P})$.
(2) If $h_1, \dots, h_n$ are elements of $\mathcal{S}$, then

$$d\gamma(h_1, \dots, h_n) = \sum_{j=1}^{n-1} \overline{\gamma(h_1, \dots, h_j)} \gamma(h_{j+1}, \dots, h_n),$$

where $\overline{a} = (-1)^{m+1} a$ for $a \in \boldsymbol{P}_m$.

Definition 2.3. Adopt the notation of (2.1). If every set of homogeneous elements of $IH(\boldsymbol{P})$ admits a trivial Massey operation, then $\boldsymbol{P}$ is a *Golod* algebra.

If $S_\bullet$ is a graded $k-$algebra, then the Poincaré series of k over $S_\bullet$ is defined to be

$$P_{S_\bullet}(z) = P^k_{S_\bullet}(z) = \sum_{i=0}^{\infty} \left(\sum_{p+q=i} \dim_k \operatorname{Tor}^{S_\bullet}_{pq}(k,k) \right) z^i. \tag{2.4}$$

(More discussion of the bigraded module $\operatorname{Tor}^{S_\bullet}(k,k)$ may be found at the beginning of [2] or [15].)

Theorem 2.5. ([3, Theorem 2.3]) *If the notation of (2.1) is adopted, the following statements are equivalent.*

1. *The $DG\Gamma-$algebra $\boldsymbol{P}$ is Golod.*
2. *The Poincaré series $P_{\boldsymbol{P}}(z)$ is equal to $\left(1 - z\sum_{i=1}^{\infty} \dim_k H_i(\boldsymbol{P}) z^i\right)^{-1}$.* $\square$

Lemma 2.6. ([8, Lemma 5.7]) *Adopt the notation of (2.1). If there is exists a $\boldsymbol{P}_0-$module V contained in $I\boldsymbol{P}$ with $IZ(\boldsymbol{P}) \subseteq V + B(\boldsymbol{P})$ and $V^2 \subseteq dV$, then $\boldsymbol{P}$ is a Golod algebra.* $\square$

The next result is a modified version of Example 5.9 in [8].

Corollary 2.7. *Let $(S_\bullet, d)$ be a $DG\Gamma-$algebra which satisfies the hypotheses of (2.1) with $S_0 = k$ and d identically zero. Suppose that there exist linearly independent elements $x_1, \dots, x_m$ in S_1 and an integer r, with $1 \le r \le m+1$, such that $S_\bullet = \overline{E} \ltimes L$, where*

(a) *$E = \bigoplus_{i=0}^{m} E_i$ is the exterior algebra $\bigwedge^\bullet(\bigoplus_{i=1}^{m} kx_i)$,*
(b) *$\overline{E} = E/E_{r+1}$,*
(c) *$L = \bigoplus_{i\ge 1} L_i$ is an $\overline{E}-$module, and*
(d) *$E_r L = 0$.*

Then the $DG\Gamma-$algebra $\boldsymbol{P} = S_\bullet{<}X_1, \dots, X_m; dX_i = x_i{>}$ is a Golod algebra.

Proof. If N is a subspace of the vector space $\boldsymbol{P}$, then let $N{<}X{>}$ represent the subspace

$$N{<}X{>} = \left\{ \sum n_{\mathbf{a}} X_1^{(a_1)} \cdots X_m^{(a_m)} \mid n_{\mathbf{a}} \in N \right\} \tag{2.8}$$

of the vector space $\boldsymbol{P}{<}X_1, \dots, X_m{>}$. Define V to be the subspace $(E_r \oplus L){<}X{>}$ of $\boldsymbol{P}$. The hypothesis ensures that $V^2 = 0$. If $z \in IZ(\boldsymbol{P})$, then $z = v + u$ for some $v \in V$ and some $u \in (\bigoplus_{i=0}^{r-1} E_i){<}X{>}$. Apply the differential d to the cycle z in order to see that

$$du = -dv \in \left(\overline{E}{<}X{>}\right) \cap (L{<}X{>}) = 0.$$

It follows that u is a cycle in $\boldsymbol{P}$. The complex $\overline{E}{<}X{>}$ of $\boldsymbol{P}$ is a homomorphic image of the acyclic complex $E{<}X{>}$; therefore, $u \in d\left((\bigoplus_{i=0}^{r-2} E_i){<}X{>}\right)$, $IZ(\boldsymbol{P}) \subseteq V + B(\boldsymbol{P})$, and the proof is complete by Lemma 2.6. $\square$

Example 2.9. *Let $S_\bullet$ be one of the $k-$algebras from Table 1 and let $W = \bigoplus_{i\geq 1} W_i$ be a trivial $S_\bullet-$module with $\dim_k W_i < \infty$ for all i. If $\boldsymbol{P}$ is the divided polynomial algebra defined below, then the DG$\Gamma-$algebra $\boldsymbol{P} \otimes_{S_\bullet} (S_\bullet \ltimes W)$ is a Golod algebra.*

$S_\bullet$	$\boldsymbol{P}$
$\mathbf{C}[p], \mathbf{C}^{(2)}, \mathbf{C}^\star$	$S_\bullet{<}X_1, X_2; d(X_i) = x_i{>}$
$\mathbf{A}, \mathbf{B}[p], \mathbf{D}[p], \mathbf{D}^{(2)}, \mathbf{E}[p], \mathbf{E}^{(q)}$	$S_\bullet{<}X_1, X_2, X_3; d(X_i) = x_i{>}$
$\mathbf{F}[p], \mathbf{F}^{(2)}, \mathbf{F}^{(3)}, \mathbf{F}^{(4)}$	$S_\bullet{<}X_1, X_2, X_3, X_4; d(X_i) = x_i{>}$
$\mathbf{F}^{(5)}$, or $\mathbf{F}^\star$ with $\operatorname{char} k = 2$	$S_\bullet{<}X_1, X_2, X_3, X_4, X_5; d(X_i) = x_i{>}$
$\mathbf{F}^\star$ with $\operatorname{char} k \neq 2$	$S_\bullet{<}X_1, X_2, X_3, X_4, X_5, Y_1; d(X_i) = x_i, d(Y_1) = y_1{>}$

Proof. We first assume that $S_\bullet \neq \mathbf{F}^\star$, or else that $S_\bullet = \mathbf{F}^\star$ and $\operatorname{char} k = 2$. Let m and r be the integers given in the following table.

(2.10)

$S_\bullet$	m	r
$\mathbf{C}[p], \mathbf{C}^{(2)}, \mathbf{C}^\star$	2	3
$\mathbf{A}, \mathbf{B}[p], \mathbf{D}[p], \mathbf{D}^{(2)}, \mathbf{E}[p], \mathbf{E}^{(q)}$	3	3
$\mathbf{F}[p], \mathbf{F}^{(2)}, \mathbf{F}^{(3)}, \mathbf{F}^{(4)}$	4	2
$\mathbf{F}^{(5)}$, or $\mathbf{F}^\star$ with $\operatorname{char} k = 2$	5	2

For $S_\bullet \neq \mathbf{F}^\star$, the result follows directly from Corollary 2.7. If $S_\bullet = \mathbf{F}^\star$ and $\operatorname{char} k = 2$, then Corollary 2.7 does not apply because y_1 and y_2 are in L but $y_1 y_2 \neq 0$. On the other hand,

$$y_1 y_2 X_1^{(a_1)} X_2^{(a_2)} \cdots X_5^{(a_5)} = x_1 z_6 X_1^{(a_1)} X_2^{(a_2)} \cdots X_5^{(a_5)} = d\left(z_6 X_1^{(a_1+1)} X_2^{(a_2)} \cdots X_5^{(a_5)}\right) \in dV.$$

A slight modification of Corollary 2.7 yields the result.

We now take $S_\bullet$ to be $\mathbf{F}^\star$ with $\operatorname{char} k \neq 2$. Let $\boldsymbol{P}' = \boldsymbol{P} \otimes_{S_\bullet} (S_\bullet \ltimes W)$, and let M_1, M_2, and N be the subspaces

$$\begin{aligned} M_1 &= (1, x_1, x_2, x_3, x_4, x_5, y_1), \\ M_2 &= (1, x_1, x_2, x_3, x_4, x_5, y_1, y_3, \dots y_{12}, z_1, \dots, z_5, w_2), \quad \text{and} \\ N &= (y_2, y_3, \dots, y_{12}, z_1, \dots, z_{10}, w_1, w_2) \oplus W \end{aligned}$$

of $S_\bullet \ltimes W$. Recall the notation of (2.8), and let $U = M_1{<}X, Y{>}$ and $V = N{<}X, Y{>}$ be subspaces of $\boldsymbol{P}'$. Observe that $V^2 = 0$. It is clear that $V \oplus U = \boldsymbol{P}'$ (as vector spaces). If $z \in IZ(\boldsymbol{P}')$, then $z = v + u$ for some $v \in V$ and some $u \in U$. Apply d to z in order to see that

$$du = -dv \in (M_2{<}X, Y{>}) \cap ((w_1){<}X, Y{>}) = 0.$$

It follows that u is a cycle in $\boldsymbol{P}'$. We proceed as in the proof of Corollary 2.7. The DG$-$algebra

$$Q = \left(\bigwedge_k^{\bullet} (\oplus_{i=1}^5 k x_i) \otimes_k \operatorname{Sym}_\bullet^k (k y_1) \right) {<}X, Y{>}$$

is known to be acyclic; see, for example, [16, Theorem 5.2]. Furthermore, the complex $M_2{<}X,Y{>}$ is a homomorphic image of Q. We conclude that $u \in d(k{<}X,Y{>})$ and $\boldsymbol{P}'$ is a Golod algebra by Lemma 2.6. □

Remarks. (a) We established Example 2.9 by identifying a subspace V of $\boldsymbol{P}$ which contains a representative of every nonzero element of $IH(\boldsymbol{P})$. A more detailed description of the homology of $\boldsymbol{P}$ is given in the proof of Lemma 3.2; consequently, an alternate proof of Example 2.9 may be read from Table 4.
(b) The behavior of the DG$\Gamma-$algebra $\mathbf{F}^\star$ depends on char k because $y_1^2 = 2y_1^{(2)} = 2w_2$. If char $k = 2$, then $y_1^2 = 0$. If char $k \neq 2$, then y_1^2 is part of a basis for $\mathbf{F}^\star$ over k.

Section 3. The list of Poincaré series.

If M is a finitely generated module over a local ring A, then the Poincaré series $P_A^M(z)$ is defined at the beginning of the paper. We write $P_A(z)$ to mean $P_A^k(z)$. The Poincaré series $P_A(z)$ is not always a rational function [1]; however, Theorem 3.3 supplies a sufficient condition for this conclusion.

The problem of computing Poincaré series may sometimes be converted from the category of local rings to the category of finite dimensional algebras over a field. If $S_\bullet$ is a graded $k-$algebra, then the Poincaré series $P_{S_\bullet}(z)$ is defined in (2.4). To compute the Poincaré series of codimension four almost complete intersections, we use Avramov's Theorem.

Theorem 3.1. ([2, Corollary 3.3]) *Let J be a small ideal in the local ring $(R, \mathfrak{M}, k)$, $A = R/J$, and $T_\bullet = \operatorname{Tor}_\bullet^R(A,k)$. If the minimal resolution of A by free $R-$modules is a DG$\Gamma-$algebra, then $P_A(z) = P_R(z)P_{T_\bullet}(z)$.* □

Recall that an ideal J in a local ring $(R, \mathfrak{M}, k)$ is said to be *small* if the natural map $\operatorname{Tor}_\bullet^R(k,k) \to \operatorname{Tor}_\bullet^{R/J}(k,k)$ is an injection. For example, if R is regular and $J \subseteq \mathfrak{M}^2$, then J is small; see [2, Example 3.11] or [15, Example 1.6].

Lemma 3.2. *Let $T_\bullet$ be a DG$\Gamma-$algebra of the form $S_\bullet \ltimes W$ for some $S_\bullet$ from Table 1 and some trivial $S_\bullet-$module W. Assume that $T_\bullet = \bigoplus_{i=0}^4 T_i$ with $T_0 = k$, $\dim_k T_1 = 5$, $\dim_k T_4 = t$ (if $S_\bullet = \mathbf{C}^\star$, then take $t = 2$), and $\dim_k T_2$, and $\dim_k T_3$ given in (1.3). Then the Poincaré series $P_{T_\bullet}(z)$ is given in Table 2.*

Theorem 3.3. *Let $(R, \mathfrak{M}, k)$ be a local ring in which 2 is a unit, J be a grade four almost complete intersection ideal in R, and $A = R/J$. If the ideal J of R is small (for example, if R is regular and $J \subseteq \mathfrak{M}^2$), then $P_A(z) = P_R(z)P_{T_\bullet}(z)$, where the Poincaré series $P_{T_\bullet}(z)$ is given in Lemma 3.2.*

Proof of Theorem 3.3. Inflate the residue field of R [12, $0_{\mathrm{III}}10.3.1$], if necessary, in order to assume that k is closed under square roots. Theorem 1.1 (together with Observation 1.5) shows that $T_\bullet = \operatorname{Tor}_\bullet^R(A,k)$ satisfies the hypotheses of Lemma 3.2; and therefore, the Poincaré series $P_{T_\bullet}(z)$ is given in Table 2. The minimal $R-$resolution of A is a DG$\Gamma-$algebra (the DG structure is exhibited in [18] and the divided powers are given by $a^{(2)} = (1/2)a^2$ for all homogeneous a of degree two); and therefore, the result follows from Theorem 3.1. □

The list of Poincaré series for Lemma 3.2

$S_\bullet$	$P_{T_\bullet}^{-1}(z)$
$\mathbf{A}$	$(1-2z-2z^2+(6-t)z^3-2z^4-2z^5+z^6)(1+z)^2$
$\mathbf{B}[p]$	$(1-2z-2z^2+(6-t)z^3+(p-3)z^4-z^5+z^6)(1+z)^2$
$\mathbf{C}[p]$	$(1-2z-2z^2+(6-t)z^3+(p-3)z^4)(1+z)^2$
$\mathbf{C}^{(2)}$	$(1-2z-2z^2+(6-t)z^3-z^4-z^5)(1+z)^2$
$\mathbf{C}^\star$	$(1-2z-2z^2+4z^3+z^4-2z^5)(1+z)^2=(1-2z)(1-z)^2(1+z)^4$
$\mathbf{D}[p]$	$(1-2z-2z^2+(6-t)z^3+(p-4)z^4-z^5+z^6)(1+z)^2$
$\mathbf{D}^{(2)}$	$(1-2z-2z^2+(6-t)z^3-2z^4-2z^5+z^6)(1+z)^2$
$\mathbf{E}[p]$	$(1-2z-2z^2+(6-t)z^3+(p-5)z^4-2z^5+2z^6)(1+z)^2$
$\mathbf{E}^{(q)}$	$(1-2z-2z^2+(6-t)z^3+(q-5)z^4-(1+q)z^5+(4-q)z^6+(q-2)z^7)(1+z)^2$
$\mathbf{F}[p]$	$(1-2z-2z^2+(6-t)z^3+(p-6)z^4-4z^5+4z^6+z^7-z^8)(1+z)^2$
$\mathbf{F}^{(2)}$	$(1-2z-2z^2+(6-t)z^3-4z^4-5z^5+4z^6+z^7-z^8)(1+z)^2$
$\mathbf{F}^{(3)}$	$(1-2z-2z^2+(6-t)z^3-3z^4-6z^5+3z^6+2z^7-z^8)(1+z)^2$
$\mathbf{F}^{(4)}$	$(1-2z-2z^2+(6-t)z^3-2z^4-7z^5+z^6+4z^7-z^9)(1+z)^2$
$\mathbf{F}^{(5)}$ $\mathbf{F}^\star$, $\operatorname{char} k=2$	$(1-2z-2z^2+(6-t)z^3-z^4-8z^5-2z^6+7z^7+3z^8-4z^9-z^{10}+z^{11})(1+z)^2$
$\mathbf{F}^\star$, $\operatorname{char} k\neq 2$	$\dfrac{(1-2z-2z^2+(7-t)z^3-3z^4-9z^5+(3-t)z^6+2z^7-z^8)(1+z)^2}{1+z^3}$

Table 2

Proof of Lemma 3.2. Our calculation of $P_{T_\bullet}(z)$ is similar to the calculation of Table 1 in [4]; some of the steps may also be found in section one of [15]. We are given $T_\bullet = S_\bullet \ltimes W$ with W a trivial $S_\bullet$−module. It follows that

$$P_{T_\bullet}^{-1}(z) = P_{S_\bullet}^{-1}(z) - z\left(\sum_{i=1}^{4} \dim_k W_i\, z^i\right). \tag{3.4}$$

Read the dimension of each W_i from (1.3) in order to obtain Table 3.

The Poincaré series $P_{\mathbf{A}}^{-1}(z) = (1-z^2)^4 - z^6$ may be read from Example 1.1 and Theorem 1.4 in [15]. The decompositions

$$\begin{aligned}
\mathbf{B}[p] &= \left(\frac{k[x_2,x_3,x_4]}{(x_2x_3x_4)} \ltimes \Big(k(-1)\oplus k(-2)^p \oplus k(-3)^p\Big)\right) \otimes_k k[x_1],\\
\mathbf{C}[p] &= \left(\left(\frac{k[x_3,x_4,x_5]}{(x_3,x_4,x_5)^2} \otimes_k k[x_2]\right) \ltimes \Big(k(-2)^p \oplus k(-3)^p\Big)\right) \otimes_k k[x_1], \quad \text{and}\\
\mathbf{C}^\star &= \left(\frac{k[x_3,x_4,x_5,y_1,y_2]}{(x_3,x_4,x_5,y_1,y_2)^2}\right) \otimes_k k[x_1,x_2]
\end{aligned}$$

have been observed in [17]. It follows that

$$\begin{aligned}
P_{\mathbf{B}[p]}^{-1}(z) &= \left((1-z^2)^3 - z^5 - z\left(z+pz^2+pz^3\right)\right)(1-z^2),\\
P_{\mathbf{C}[p]}^{-1}(z) &= \left((1-3z^2)(1-z^2) - z(pz^2+pz^3)\right)(1-z^2), \quad \text{and}\\
P_{\mathbf{C}^\star}^{-1}(z) &= \left(1 - z(3z+2z^2)\right)(1-z^2)^2.
\end{aligned}$$

The trivial $S_\bullet$−module W

$S_\bullet$	$\sum_{i=1}^4 \dim_k W_i\, z^i$
$\mathbf{A}$	$z + tz^2(1+z)^2 - 2z^3$
$\mathbf{B}[p]$	$(t-p)z^2(1+z)^2$
$\mathbf{C}[p]$	$(t-p)z^2(1+z)^2$
$\mathbf{C}^{(2)}$	$(t-1)z^2(1+z)^2 - 2z^3$
$\mathbf{C}^\bigstar$	0
$\mathbf{D}[p]$	$(t-p)z^2(1+z)^2 + 2z^3$
$\mathbf{D}^{(2)}$	$(t-1)z^2(1+z)^2$
$\mathbf{E}[p]$	$(t-p)z^2(1+z)^2 + 4z^3$
$\mathbf{E}^{(q)}$	$(t-1)z^2(1+z)^2 + (6-2q)z^3$
$\mathbf{F}[p]$	$(t-p)z^2(1+z)^2 + 6z^3$
$\mathbf{F}^{(r)}$	$(t-1)z^2(1+z)^2 + (8-2r)z^3$
$\mathbf{F}^\bigstar$	$(t-2)z^2(1+z)^2$

Table 3

For any other choice of $S_\bullet$, let $\boldsymbol{P}$ be the Golod DGΓ−algebra defined in Example 2.9, and let $F_{\boldsymbol{P}}(z)$ be the formal power series

$$F_{\boldsymbol{P}}(z) = \sum_{i=1}^{\infty} \dim_k H_i(\boldsymbol{P}) z^i.$$

Theorem 2.5 shows that $P_{\boldsymbol{P}}^{-1}(z) = 1 - zF_{\boldsymbol{P}}(z)$; consequently,
(3.5)
$$P_{S_\bullet}^{-1}(z) = \begin{cases} (1-z^2)^m(1-zF_{\boldsymbol{P}}(z)), & \text{for } S_\bullet \neq \mathbf{F}^\bigstar, \text{ or } S_\bullet = \mathbf{F}^\bigstar \text{ with } \operatorname{char} k = 2, \\ \frac{(1-z^2)^5}{(1+z^3)}(1-zF_{\boldsymbol{P}}(z)), & \text{for } S_\bullet = \mathbf{F}^\bigstar \text{ with } \operatorname{char} k \neq 2, \end{cases}$$

where m is given in (2.10).

In order to compute the homology of $\boldsymbol{P}$, we decompose the subcomplex

$$(3.6) \qquad C_n : \quad \boldsymbol{P}_{2n+2} \xrightarrow{d_{2n+2}} \boldsymbol{P}_{2n+1} \xrightarrow{d_{2n+1}} \boldsymbol{P}_{2n} \xrightarrow{d_{2n}} \boldsymbol{P}_{2n-1},$$

for $n \geq 0$, into a direct sum of smaller complexes. The following notation is in effect throughout this discussion. Let $X^{(q)}$ represent the subspace of the vector space $\boldsymbol{P}$ which consists of all k−linear combinations of the divided power monomials $X_1^{(a_1)} \cdots X_m^{(a_m)}$, where $\sum a_i = q$. If $s_1, \ldots, s_p \in S_\bullet$, then let $(s_1, \ldots, s_p)$ be the subspace of $\boldsymbol{P}$ spanned by all k−linear combinations of $s_1, \ldots, s_p$. If A and B are subspaces of $\boldsymbol{P}$, then AB is the subspace of $\boldsymbol{P}$ spanned by $\{ab \mid a \in A \text{ and } b \in B\}$.

Now we consider $S_\bullet = \mathbf{C}^{(2)}$. Let M be the subspace $(x_1, x_2)(x_3, x_4, x_5)$ of $S_\bullet$.

The complex C_n is the direct sum of the following complexes.

$$\begin{array}{lccccccc}
C_{n,1}: & (1)X^{(n+1)} & \to & (x_1,x_2)X^{(n)} & \to & (x_1x_2)X^{(n-1)} & \to & 0 \\
C_{n,2}: & 0 & \to & (x_3,x_4,x_5)X^{(n)} & \to & MX^{(n-1)} & \to & (x_1x_2)(x_3,x_4,x_5)X^{(n-2)} \\
C_{n,3}: & 0 & \to & 0 & \to & (y_1)X^{(n-1)} & \to & (x_1,x_2)(y_1)X^{(n-2)} \\
C_{n,4}: & 0 & \to & 0 & \to & (1)X^{(n)} & \to & (x_1,x_2)X^{(n-1)} \\
C_{n,5}: & (y_1)X^{(n)} & \to & (x_1,x_2)(y_1)X^{(n-1)} & \to & 0 & \to & 0 \\
C_{n,6}: & 0 & \to & (z_3,z_4)X^{(n-1)} & \to & (w_1)X^{(n-2)} & \to & 0 \\
C_{n,7}: & MX^{(n)} & \to & (x_1x_2)(x_3,x_4,x_5)X^{(n-1)} & \to & 0 & \to & 0 \\
C_{n,8}: & (x_1,x_2)X^{(n)} & \to & 0 & \to & 0 & \to & (x_3,x_4,x_5)X^{(n-1)} \\
C_{n,9}: & (w_1)X^{(n-1)} & \to & 0 & \to & 0 & \to & (z_3,z_4)X^{(n-2)}
\end{array}$$

The complex $C_{n,1}$ is exact because the subalgebra $k[x_1,x_2]<X_1,X_2>$ of $\boldsymbol{P}$ is acyclic. If $n=0$, then $C_{n,2}$ contributes $[x_3]$, $[x_4]$, and $[x_5]$ to $H_1(\boldsymbol{P})$. If $n\ge 1$, then $C_{n,2}$ is isomorphic to the direct sum of three copies of $C_{n-1,1}$ and is therefore exact. If $n=1$, then $C_{n,3}$ contributes $[y_1]$ to $H_2(\boldsymbol{P})$. If $n\ge 2$, then $C_{n,3}$ is exact. We see that $C_{n,i}$ is exact for i is equal to 4, 7, 8, or 9. If $n\ge 1$, then the homology at $(x_1,x_2)y_1X^{(n-1)}$ in $C_{n,5}$ has dimension $2n-(n+1)$ and the homology at $(z_3,z_4)X^{(n-1)}$ in $C_{n,6}$ has dimension $2n-(n-1)$. Thus,

$$\dim_k H_{2n+1}(\boldsymbol{P}) = \begin{cases} 2n, & \text{if } 1\le n, \\ 3, & \text{if } 0=n, \end{cases} \quad\text{and}\quad \dim_k H_{2n}(\boldsymbol{P}) = \begin{cases} 0, & \text{if } 2\le n, \text{ and} \\ 1, & \text{if } 1=n. \end{cases}$$

The equality

$$\sum_{n=a-b}^{\infty}\binom{n+b}{a}z^{2n} = \frac{z^{2(a-b)}}{(1-z^2)^{a+1}}, \tag{3.7}$$

for integers a and b with $a\ge 0$, is well known. It follows that

$$F_{\boldsymbol{P}}(z) = 3z+z^2+\sum_{n=1}^{\infty}2nz^{2n+1} = 3z+z^2+\frac{2z^3}{(1-z^2)^2}.$$

An analogous decomposition of (3.6) can be made for each of the other choices of $S_\bullet$. In Table 4 we record where the homology of $\boldsymbol{P}$ lives without explicitly recording the decomposition of C_n. The details have been omitted, except, as an example, we have recorded three of the summands of C_n in the most complicated case; that is, when $S_\bullet = \mathbf{F}^\star$ and $\operatorname{char} k \ne 2$. It is easy to see that the map d_{2n+1} is surjective in the complex

$$C_{n,1}: \quad 0 \xrightarrow{d_{2n+2}} \begin{array}{c}(z_6,\dots,z_{10})X^{(n-1)}\\ \bigoplus \\ (y_2)(Y_1)X^{(n-2)}\end{array} \xrightarrow{d_{2n+1}} (w_1)X^{(n-2)} \xrightarrow{d_{2n}} 0;$$

consequently, all of the homology in this complex is concentrated in position $2n+1$. The complex

$$C_{n,2}: \quad 0 \xrightarrow{d_{2n+2}} (Y_1)X^{(n-1)} \xrightarrow{d_{2n+1}} \begin{array}{c}(y_1)X^{(n-1)}\\ \bigoplus \\ (x_1,\dots,x_5)(Y_1)X^{(n-2)}\end{array} \xrightarrow{d_{2n}} \begin{array}{c}(x_1,\dots x_5)(y_1)X^{(n-2)}\\ \bigoplus \\ (x_1,\dots x_5)^2(Y_1)X^{(n-3)}\end{array}$$

is exact. The complex

$$C_{n,3}: \quad \begin{matrix} (y_1)X^{(n)} \\ \bigoplus \\ (x_1,\dots,x_5)(Y_1)X^{(n-1)} \end{matrix} \xrightarrow{d_{2n+2}} \begin{matrix} (x_1,\dots x_5)(y_1)X^{(n-1)} \\ \bigoplus \\ (x_1,\dots x_5)^2(Y_1)X^{(n-2)} \end{matrix} \xrightarrow{d_{2n+1}} 0 \xrightarrow{d_{2n}} 0$$

is the tail end of the exact complex $C_{n+1,2}$; consequently, it is easy to compute the homology at position $2n+1$.

A routine calculation using Table 4 and (3.7) produces the power series $F_{\boldsymbol{P}}(z)$; the result is recorded in Table 5. The proof is completed by combining Table 5 with (3.5), (3.4), and Table 3. □

Example 3.8. Let $(R,\mathfrak{M},k)$ be a regular local ring. Suppose that $Y_{1\times 5}$ and $X_{5\times 5}$ are matrices with entries in $\mathfrak{M}$, with X alternating. Assume that the ideal $J = I_1(YX)$ has grade four. Let $A = R/J$ and $T_\bullet = \operatorname{Tor}_\bullet^R(A,k)$. One can compute that $T_\bullet = \mathbf{F}^\star$ for any field k. If $\operatorname{char} k \neq 2$, then Theorem 3.3 shows that

$$P_A(z) = \frac{(1+z^3)P_R(z)}{(1-2z-2z^2+5z^3-3z^4-9z^5+z^6+2z^7-z^8)(1+z)^2}.$$

On the other hand, the techniques of the present paper can be used to calculate the Poincaré series $P_A(z)$ even if $\operatorname{char} k = 2$. One can show that the minimal $R-$resolution of A is a DG$\Gamma-$algebra; consequently, Theorem 3.1 yields that $P_A(z) = P_R(z)P_{T_\bullet}(z)$. The Poincaré series $P_{T_\bullet}(z)$ is given in Table 2; and therefore,

$$P_A(z) = \frac{P_R(z)}{(1-2z-2z^2+4z^3-z^4-8z^5-2z^6+7z^7+3z^8-4z^9-z^{10}+z^{11})(1+z)^2}.$$

Section 4. The Poincaré series of modules.

In Theorem 3.3 we proved that the Poincaré series $P_A^k(z)$ is a rational function whenever $(A,\mathfrak{m},k)$ is a codimension four almost complete intersection in which two is a unit. In the present section, we apply Theorem 4.1, which is a new result due to Avramov, in order to conclude that $P_A^M(z)$ is a rational function for all finitely generated $A-$modules M.

Theorem 4.1 refers to data from two Tate resolutions. If $(A,\mathfrak{m},k)$ is a local ring, then the Tate resolution X of k over A is the DG$\Gamma-$algebra which is the union of the following collection of DG$\Gamma-$subalgebras

$$A = X(0) \subseteq X(1) \subseteq X(2) \subseteq \dots \quad .$$

Each $X(n)$ has the form

$$X(n) = X(n-1){<}Y_1,\dots,Y_{e_n}; d(Y_i) = z_i{>},$$

where each Y_i is a divided power variable of degree n and $z_1,\dots z_{e_n}$ are cycles in $X(n-1)$ which represent a minimal generating set for the kernel of $H_{n-1}(X(n-1)) \to H_{n-1}(k)$. (In the above discussion we have viewed A and k as graded algebras

The homology in $\boldsymbol{P}$

$S_\bullet$	the homology in $\boldsymbol{P}$ at	has dimension	$H_i(\boldsymbol{P})$
$\mathbf{D}[p]$	$(x_4,x_5)X^{(n)}$ $(z_1,\dots,z_p)X^{(n-1)}$	2 if $n=0$, 1 if $n\ge 1$ $p\binom{n+1}{2}-p\binom{n}{2}$	$2n+1$
	$(x_1x_4,x_1x_5,x_2x_4,x_2x_5,x_3x_4)X^{(n-1)}$ $(y_1,\dots,y_p)X^{(n-1)}$	$5\binom{n+1}{2}-\binom{n}{2}-2\binom{n+2}{2}+1$ $p\binom{n+1}{2}-p\binom{n}{2}$	$2n$
$\mathbf{D}^{(2)}$	$(x_4,x_5)X^{(n)}$ $(x_1,x_2)(y_1)X^{(n-1)}$ $(z_3,z_4)X^{(n-1)}$	2 if $n=0$, 1 if $n\ge 1$ $2\binom{n+1}{2}-\binom{n+2}{2}+1$ $2\binom{n+1}{2}-\binom{n}{2}$	$2n+1$
	$(x_1x_4,x_1x_5,x_2x_4,x_2x_5,x_3x_4)X^{(n-1)}$ $(y_1)X^{(n-1)}$	$5\binom{n+1}{2}-\binom{n}{2}-2\binom{n+2}{2}+1$ 1	$2n$
$\mathbf{E}[p]$	$(x_4,x_5)X^{(n)}$ $(z_1,\dots,z_p)X^{(n-1)}$	2 if $n=0$, 0 if $n\ge 1$ $p\binom{n+1}{2}-p\binom{n}{2}$	$2n+1$
	$(x_1,x_2,x_3)(x_4,x_5)X^{(n-1)}$ $(y_1,\dots,y_p)X^{(n-1)}$	$6\binom{n+1}{2}-2\binom{n+2}{2}$ $p\binom{n+1}{2}-p\binom{n}{2}$	$2n$
$\mathbf{E}^{(q)}$	$(x_4,x_5)X^{(n)}$ $(x_1,\dots,x_q)(y_1)X^{(n-1)}$ $(z_{q+1},\dots,z_{2q})X^{(n-1)}$	2 if $n=0$, 0 if $n\ge 1$ 0 if $n=0$, $q\binom{n+1}{2}-\binom{n+2}{2}+(3-q)$ if $n\ge 1$ $q\binom{n+1}{2}-\binom{n}{2}$	$2n+1$
	$(x_1,x_2,x_3)(x_4,x_5)X^{(n-1)}$ $(y_1)X^{(n-1)}$	$6\binom{n+1}{2}-2\binom{n+2}{2}$ 1 if $n=1$, $3-q$ if $n\ge 2$	$2n$
$\mathbf{F}[p]$	$(x_5)X^{(n)}$ $(z_1,\dots,z_p)X^{(n-1)}$	1 if $n=0$, 0 if $n\ge 1$ $p\binom{n+2}{3}-p\binom{n+1}{3}$	$2n+1$
	$(x_1,x_2,x_3,x_4)^2X^{(n-1)}$ $(x_1,x_2,x_3,x_4)(x_5)X^{(n-1)}$ $(y_1,\dots,y_p)X^{(n-1)}$	$6\binom{n+2}{3}-4\binom{n+3}{3}+\binom{n+4}{3}$ $4\binom{n+2}{3}-\binom{n+3}{3}$ $p\binom{n+2}{3}-p\binom{n+1}{3}$	$2n$
$\mathbf{F}^{(2)}$	$(x_5)X^{(n)}$ $(x_1,x_2)(y_1)X^{(n-1)}$ $(z_3,z_4)X^{(n-1)}$	1 if $n=0$, 0 if $n\ge 1$ $2\binom{n+2}{3}-\binom{n+3}{3}+n+1$ $2\binom{n+2}{3}-\binom{n+1}{3}$	$2n+1$
	$(x_1,x_2,x_3,x_4)^2X^{(n-1)}$ $(x_1,x_2,x_3,x_4)(x_5)X^{(n-1)}$ $(y_1)X^{(n-1)}$	$6\binom{n+2}{3}-4\binom{n+3}{3}+\binom{n+4}{3}$ $4\binom{n+2}{3}-\binom{n+3}{3}$ n	$2n$

KEY: The second row of this table should be read, "If $S_\bullet=\mathbf{D}[p]$, then the homology in $\boldsymbol{P}$ at $(z_1,\dots,z_p)X^{(n-1)}$ has dimension $p\binom{n+1}{2}-p\binom{n}{2}$; furthermore, this homology contributes to $H_{2n+1}(\boldsymbol{P})$."

Table 4 — Part 1

concentrated in degree zero.) In particular, $e_1 = \dim_k \mathfrak{m}/\mathfrak{m}^2$. Furthermore, if $A = R/I$ where $(R, \mathfrak{M}, k)$ is regular local and $I \subseteq \mathfrak{M}^2$, then $e_2 = \dim_k \operatorname{Tor}_1^R(A,k)$; in other words, $e_2 = \dim_k(I/\mathfrak{M}I)$. If $T_\bullet$ is the algebra $\operatorname{Tor}_\bullet^R(A,k)$, then the Tate resolution $\widetilde{X}$ of k over $T_\bullet$ is obtained in a similar manner; see [16] for details. Indeed, $\widetilde{X}$ is the union of the DG$\Gamma-$subalgebras

$$T_\bullet = \widetilde{X}(0) = \widetilde{X}(1) \subseteq \widetilde{X}(2) \subseteq \widetilde{X}(3) \subseteq \dots \quad ,$$

where each $\widetilde{X}(n)$ has the form

$$\widetilde{X}(n) = \widetilde{X}(n-1)<Y_1, \dots, Y_{\widetilde{e}_n}>,$$

and each Y_i is a divided power variable of degree n. If the minimal resolution of A by free $R-$modules is a DG$\Gamma-$algebra, then Theorem 3.1 shows that $\widetilde{e}_n = e_n$ for $2 \le n$.

Theorem 4.1. ([6]) *Let $(R, \mathfrak{M}, k)$ be a regular local ring, $I \subseteq \mathfrak{M}^2$ be an ideal of R, A be the quotient R/I, $T_\bullet$ be the algebra $\operatorname{Tor}_\bullet^R(A,k)$, and $\widetilde{X}$ be the minimal Tate resolution of k over $T_\bullet$. Assume that the minimal resolution of A by free $R-$modules is a DG$\Gamma-$algebra and that there exists an integer n and divided power variables $Y_1, \dots, Y_s$ of degree n such that the DG$\Gamma-$subalgebra $\widetilde{X}(n-1)<Y_1, \dots, Y_s>$ of $\widetilde{X}$ is Golod. Then the Poincaré series $P_A^M(z)$ is a rational function for all finitely generated $A-$modules M. In fact, there is a polynomial $\operatorname{Den}_A(z) \in \mathbb{Z}[z]$ with*

(a) $$P_A(z) = \frac{(1+z)^{e_1}(1+z^3)^{e_3}\cdots(1+z^{m-2})^{e_{m-2}}(1+z^m)^r}{\operatorname{Den}_A(z)},$$

$$\text{where} \quad \begin{cases} m = n \text{ and } r = s, & \text{if } n \text{ is odd,} \\ m = n-1 \text{ and } r = e_{n-1}, & \text{if } n \text{ is even,} \end{cases} \quad \text{and}$$

(b) $\operatorname{Den}_A(z) P_A^M(z) \in \mathbb{Z}[z]$ *for all finitely generated $A-$modules M.* □

Corollary 4.2. *Let $(R, \mathfrak{M}, k)$ be a regular local ring, and $(A, \mathfrak{m}, k)$ be the quotient R/J, where J is an almost complete intersection ideal of grade at most four. If two is a unit in A, then there is a polynomial $\operatorname{Den}_A(z) \in \mathbb{Z}[z]$ such that $\operatorname{Den}_A(z) P_A^M(z) \in \mathbb{Z}[z]$ for all finitely generated $A-$modules M.*

Proof. The Betti numbers of M are unchanged under a faithfully flat extension of A; consequently, we may assume that k is closed under square roots. We may replace R by $R/(x)$ for some $x \in \mathfrak{M} \setminus \mathfrak{M}^2$, if necessary, in order to assume that $J \subseteq \mathfrak{M}^2$. Let g represent the grade of J, $T_\bullet = \operatorname{Tor}_\bullet^R(A,k)$, and $t = \dim_k(T_g)$. If $g \le 3$, then the result is contained in [8]. For the sake of completeness, we recall that $\operatorname{Den}_A(z)$ is defined by

$$\operatorname{Den}_A(z) = \begin{cases} (1+z)^2(1-2z), & \text{if } g = 2, \\ (1+z)^3(1-z)(1-2z), & \text{if } g = 3 \text{ and } t = 2, \\ (1+z)(1-z-3z^2-(t-3)z^3-z^5), & \text{if } g = 3 \text{ and } t \ge 3 \text{ is odd, and} \\ (1+z)(1-z-3z^2-(t-3)z^3), & \text{if } g = 3 \text{ and } t \ge 4 \text{ is even.} \end{cases}$$

The homology in $\boldsymbol{P}$

$S_\bullet$	the homology in $\boldsymbol{P}$ at	has dimension	$H_i(\boldsymbol{P})$
$\mathbf{F}^{(3)}$	$(x_5)X^{(n)}$	1 if $n=0$, 0 if $n\ge 1$	$2n+1$
	$(x_1,x_2,x_3)(y_1)X^{(n-1)}$	$3\binom{n+2}{3}-\binom{n+3}{3}+1$	
	$(z_4,z_5,z_6)X^{(n-1)}$	$3\binom{n+2}{3}-\binom{n+1}{3}$	
	$(x_1,x_2,x_3,x_4)^2X^{(n-1)}$	$6\binom{n+2}{3}-4\binom{n+3}{3}+\binom{n+4}{3}$	$2n$
	$(x_1,x_2,x_3,x_4)(x_5)X^{(n-1)}$	$4\binom{n+2}{3}-\binom{n+3}{3}$	
	$(y_1)X^{(n-1)}$	1	
$\mathbf{F}^{(4)}$	$(x_5)X^{(n)}$	1 if $n=0$, 0 if $n\ge 1$	$2n+1$
	$(x_1,x_2,x_3,x_4)(y_1)X^{(n-1)}$	0 if $n=0$, $4\binom{n+2}{3}-\binom{n+3}{3}$ if $n\ge 1$	
	$(z_5,z_6,z_7,z_8)X^{(n-1)}$	$4\binom{n+2}{3}-\binom{n+1}{3}$	
	$(x_1,x_2,x_3,x_4)^2X^{(n-1)}$	$6\binom{n+2}{3}-4\binom{n+3}{3}+\binom{n+4}{3}$	$2n$
	$(x_1,x_2,x_3,x_4)(x_5)X^{(n-1)}$	$4\binom{n+2}{3}-\binom{n+3}{3}$	
	$(y_1)X^{(n-1)}$	1 if $n=1$, 0 if $n\ge 2$	
$\mathbf{F}^{(5)}$	$(x)(y_1)X^{(n-1)}$	0 if $n=0$, $5\binom{n+3}{4}-\binom{n+4}{4}$ if $n\ge 1$	$2n+1$
	$(z_6,\dots,z_{10})X^{(n-1)}$	$5\binom{n+3}{4}-\binom{n+2}{4}$	
	$(x)^2X^{(n-1)}$	$10\binom{n+3}{4}-5\binom{n+4}{4}+\binom{n+5}{4}$	$2n$
	$(y_1)X^{(n-1)}$	1 if $n=1$, 0 if $n\ge 2$	
$\mathbf{F}^\star$	$(x)(y_1)X^{(n-1)}$	0 if $n=0$, $5\binom{n+3}{4}-\binom{n+4}{4}$ if $n\ge 1$	$2n+1$
char $k=2$	$(z_6,\dots,z_{10})X^{(n-1)}$	$5\binom{n+3}{4}-\binom{n+2}{4}$	
	$(x)^2X^{(n-1)}$	$10\binom{n+3}{4}-5\binom{n+4}{4}+\binom{n+5}{4}$	$2n$
	$(y_1)X^{(n-1)}$	1 if $n=1$, 0 if $n\ge 2$	
	$(y_2)X^{(n-1)}$	$\binom{n+3}{4}$	
	$(w_2)X^{(n-2)}$	$\binom{n+2}{4}$	
$\mathbf{F}^\star$	$(x)(y_1)X^{(n-1)}\oplus(x)^2(Y_1)X^{(n-2)}$	$10\binom{n+2}{4}$	$2n+1$
char $k\ne 2$	$(z_6,\dots,z_{10})X^{(n-1)}\oplus(y_2)(Y_1)X^{(n-2)}$	$5\binom{n+3}{4}$	
	$(y_1^2)(Y_1)X^{(n-3)}$	$\binom{n+1}{4}$	
	$(x)^2X^{(n-1)}$	$10\binom{n+3}{4}-5\binom{n+4}{4}+\binom{n+5}{4}$	$2n$
	$(y_1^2)X^{(n-2)}\oplus(x)(y_1)(Y_1)X^{(n-3)}$	$5\binom{n+1}{4}$	
	$(z_6,\dots,z_{10})(Y_1)X^{(n-3)}$	$5\binom{n+1}{4}-\binom{n}{4}$	
	$(y_2)X^{(n-1)}$	$\binom{n+3}{4}$	

KEY: The second row of this table should be read, "If $S_\bullet=\mathbf{F}^{(3)}$, then the homology in $\boldsymbol{P}$ at $(x_1,x_2,x_3)(y_1)X^{(n-1)}$ has dimension $3\binom{n+2}{3}-\binom{n+3}{3}+1$; furthermore, this homology contributes to $H_{2n+1}(\boldsymbol{P})$." We have written (x) to mean (x_1,x_2,x_3,x_4,x_5).

Table 4 — Part 2

The formal power series $F_{\boldsymbol{P}}(z)$

$S_\bullet$	$F_{\boldsymbol{P}}(z)$
$\mathbf{D}[p]$	$2+z+\dfrac{z}{1-z}+\dfrac{-2+(p+5)z^2+pz^3-(p+1)z^4-pz^5}{(1-z^2)^3}$
$\mathbf{D}^{(2)}$	$2+z+\dfrac{2z}{1-z}+\dfrac{-2-z+5z^2+4z^3-z^4-z^5}{(1-z^2)^3}$
$\mathbf{E}[p]$	$2+2z+\dfrac{-2+(p+6)z^2+pz^3-pz^4-pz^5}{(1-z^2)^3}$
$\mathbf{E}^{(q)}$	$2+3z+z^2+\dfrac{(3-q)z^3}{1-z}+\dfrac{-2-z+6z^2+2qz^3-z^5}{(1-z^2)^3}$
$\mathbf{F}[p]$	$-z^{-2}+1+z+\dfrac{z^{-2}-5+(10+p)z^2+pz^3-pz^4-pz^5}{(1-z^2)^4}$
$\mathbf{F}^{(r)}$ $2\le r\le 4$	$-z^{-2}+1+z+\dfrac{z+z^2}{(1-z^2)^{4-r}}+\dfrac{z^{-2}-5-z+10z^2+2rz^3-z^5}{(1-z^2)^4}$
$\mathbf{F}^{(5)}$	$-z^{-2}+z+z^2+\dfrac{z^{-2}-5-z+10z^2+10z^3-z^5}{(1-z^2)^5}$
$\mathbf{F}^\star$ char $k=2$	$-z^{-2}+z+z^2+\dfrac{z^{-2}-5-z+11z^2+10z^3+z^4-z^5}{(1-z^2)^5}$
$\mathbf{F}^\star$ char $k\neq 2$	$-z^{-2}+\dfrac{z^{-2}-5+11z^2+5z^3+10z^5+10z^6+z^7-z^8}{(1-z^2)^5}$

Table 5

Now we consider the case $g=4$. Write $T_\bullet = S_\bullet \ltimes W$, where W is a trivial $S_\bullet-$module and $S_\bullet$ is one of the algebras from Table 1. Let $\boldsymbol{P}$ be the DGΓ−defined in Example 2.9. The existence of $\operatorname{Den}_A(z)$ is guaranteed by Theorem 4.1 because $\boldsymbol{P}\otimes_{S_\bullet} T_\bullet$ is a Golod algebra. Furthermore, Theorem 4.1 also shows that $\operatorname{Den}_A(z)$ is the same as the polynomial labeled $P^{-1}_{T_\bullet}(z)$ in the statement of Theorem 3.3, unless $S_\bullet = \mathbf{F}^\star$. In the latter case

$$\operatorname{Den}_A(z) = (1-2z-2z^2+(7-t)z^3-3z^4-9z^5+(3-t)z^6+2z^7-z^8)(1+z)^2. \quad \square$$

The following application of Corollary 4.2 is proved by appealing to [14, Theorem 4.15]. Recall our convention that an almost complete intersection is never a complete intersection.

Corollary 4.3. *Let $(R,\mathfrak{M},k)$ be a regular local ring, and $(A,\mathfrak{m},k)$ be the quotient R/J, where J is an almost complete intersection ideal of grade at most four. If the field of rational numbers is contained in R, then there are infinitely many integers $i\geq 1$ for which the cotangent module $T_i(A/R,A)$ is not zero.* $\square$

Section 5. Growth of Betti numbers.

If M is a finitely generated module over a local ring $(A,\mathfrak{m},k)$, then the i^{th} Betti number of M is equal to

$$b_i = \dim_k \operatorname{Tor}_i^A(M,k).$$

The concept of the complexity of a module, which was introduced in [4, (1.1)] and [5, (3.1)], plays a crucial role in our study of Betti number growth.

Definition 5.1. Let M be a finitely generated module over a local ring $(A, \mathfrak{m}, k)$. The *complexity*, $\operatorname{cx}_A M$, of M is equal to d, if $d-1$ is the smallest degree of a polynomial $f(n) \in \mathbb{Z}[n]$ for which $b_n \le f(n)$ for all sufficiently large n. If no such d exists, then M has infinite complexity. (The zero polynomial is assigned degree -1.)

Observe that the definition of complexity is designed so that $\operatorname{cx}_A M = 0$ if and only if $\operatorname{pd}_A M < \infty$; and $\operatorname{cx}_A M = 1$ if and only if the projective dimension of M is infinite, but the Betti numbers of M are bounded.

Corollary 5.2. *Let $(R, \mathfrak{M}, k)$ be a regular local ring, and $(A, \mathfrak{m}, k)$ be the quotient R/J, where J is an almost complete intersection ideal of grade at most four. Assume that two is a unit in R. Let M be a finitely generated $A-$module of infinite projective dimension, and let b_i represent the i^{th} Betti number of M. Then one of the following cases occurs.*

1. *The Betti numbers of M grow exponentially; that is, there are real numbers α and β with $1 < \alpha \le \beta$ and $\alpha^n \le \sum_{i=0}^{n} b_i \le \beta^n$ for all large n.*
2. *The Betti numbers of M grow linearly. In this case, there are positive integers a and b with $(a/2)n - b \le b_n \le (a/2)n + b$ for all large n.*
3. *The Betti numbers of M are bounded. In this case, the minimal $A-$resolution $\mathbb{F}$ of M is eventually periodic of period at most two. In fact, $\mathbb{F}$ is eventually given by a matrix factorization; that is, there exists integers b and r, a local ring $(B, \mathfrak{n})$, an element $x \in \mathfrak{n}$, and $b \times b$ matrices ϕ and ψ, with entries in B, such that x is regular on B, $B/(x) \cong A$, $\phi\psi = xI_b = \psi\phi$, and the tail $\mathbb{F}_{\ge r}$ of $\mathbb{F}$ is given by*

$$\dots \longrightarrow A^b \xrightarrow{\overline{\psi}} A^b \xrightarrow{\overline{\phi}} A^b \xrightarrow{\overline{\psi}} A^b \xrightarrow{\overline{\phi}} A^b$$

   where $^{-}$ represents $\_\_ \otimes_B A$.

Proof. As in the proof of Corollary 4.2, we may assume that k is closed under square roots and that $J \subseteq \mathfrak{M}^2$. Let $g = \operatorname{grade} J$, $T_\bullet = \operatorname{Tor}(A, k)$, and $t = \dim_k T_g$.

If $\operatorname{cx}_A M = \infty$, then [4, Proposition 2.3] shows that the Betti numbers of M grow exponentially as described in (1). Henceforth, we assume $\operatorname{cx}_A M < \infty$. Apply Proposition 2.4 in [4] to see that $\operatorname{cx}_A M$ is the order of the pole $P_A^M(z)$ at $z = 1$. In the proof of Corollary 4.2 we identified a polynomial $\operatorname{Den}_A(z)$ with the property that $\operatorname{Den}_A(z)P_A^M(z) \in \mathbb{Z}[z]$. It follows that $\operatorname{cx}_A M$ is no more than the multiplicity of $z = 1$ as a root of $\operatorname{Den}_A(z)$. The value of $\operatorname{Den}_A(1)$ may be quickly computed. (Remember that $t \ge 2$ because A is not Gorenstein, and $t \ge p$ because of the way the algebras $\mathbf{B}[p], \dots, \mathbf{F}[p]$ are defined.) Our calculations are summarized in the following table. (The algebra $\mathbf{H}(3,2)$ was introduced in the proof of Proposition 1.4.)

$T_\bullet$	$\operatorname{cx}_A M$
$\mathbf{C}^\star$	$0 \le \operatorname{cx}_A M \le 2$
$\mathbf{B}[t]$ or $\mathbf{C}[t]$ or $\mathbf{H}(3,2)$	$0 \le \operatorname{cx}_A M \le 1$
anything else	$0 = \operatorname{cx}_A M$

The hypothesis $\operatorname{pd}_A M = \infty$ ensures that $\operatorname{cx}_A(M) \neq 0$; and therefore, $T_\bullet$ is equal to one of $\mathbf{B}[t]$, $\mathbf{C}[t]$, $\mathbf{C}^\star$, or $\mathbf{H}(3,2)$. Apply Proposition 1.4, Observation 1.5, and [4, Proposition 3.4], in order to produce an almost complete intersection $(B, \mathfrak{n}, k)$ and a regular sequence $\mathbf{a}$ such that $B/(\mathbf{a}) = A$ and $\operatorname{Den}_B(1) \neq 0$. (The ring B has the form R/J' for some almost complete intersection ideal J' with $\operatorname{grade} J' < \operatorname{grade} J$. The length of $\mathbf{a}$ is one, unless $T_\bullet = \mathbf{C}^\star$, in which case $\mathbf{a}$ has length two.) The complexity of M, as a $B-$module, is finite by (A.11) of [4]; and therefore, we may repeat the above argument in order to conclude that $\operatorname{pd}_B M < \infty$. It follows that, in the language of [5], the $A-$module M has finite virtual projective dimension. The rest of the conclusion may now be read from Theorems 4.1 and 4.4 of [5]. □

Part (3) of the above result shows that the Eisenbud conjecture holds for the rings A under consideration. Gasharov and Peeva [11] have found counterexamples to the conjecture.

Section 6. Golod homomorphisms.

Assume, for the time being, that A satisfies one of the hypotheses (a) – (d) from the beginning of the paper. It is shown in [8] that the Poincaré series $P_A^M(z)$ is rational for all finitely generated $A-$modules. The proof consists of applying Levin's Theorem (see [8, Proposition 5.18]) to a Golod homomorphism $C \twoheadrightarrow A$ for some complete intersection C. Now, take A as described in Corollary 4.2. In most cases (see Corollary 6.2 for details) one can obtain the conclusion of Corollary 4.2 by using the techniques of [8] in place of Theorem 4.1. However, if $\operatorname{Tor}_\bullet^R(A,k) = \mathbf{F}^\star$ and $\operatorname{char} k \neq 2$, then, in Proposition 6.5, we show that there does not exist a Golod map from a complete intersection onto A. In this case, we must use Theorem 4.1 in our proof of Corollary 4.2.

Definition 6.1. Let $f\colon (C, \mathfrak{n}, k) \twoheadrightarrow (A, \mathfrak{m}, k)$ be a surjection of local rings. Assume that A is not a hyperplane section of C. (In other words, A is not of the form $C/(x)$ for some regular element $x \in \mathfrak{n} \setminus \mathfrak{n}^2$.) Let X be the Tate resolution of k over C. If $A \otimes_C X$ is a Golod algebra, then f is a *Golod homomorphism.*

Corollary 6.2. *Let $(R, \mathfrak{M}, k)$ be a regular local ring in which 2 is a unit, J be a grade four almost complete intersection ideal in R, and $A = R/J$. Suppose that $\operatorname{Tor}_\bullet^R(A,k)$ has the form $S_\bullet \ltimes W$ for some $S_\bullet$ from Table 1 and some trivial $S_\bullet-$module W. (This hypothesis is satisfied if k is closed under square roots.) If $S_\bullet \neq \mathbf{F}^{(5)}$ or $\mathbf{F}^\star$, then there exists an $R-$sequence $a_1, \dots, a_m$ in J (where m is given in (2.10)), such that the natural map $R/(a_1, \dots, a_m) \twoheadrightarrow A$ is a Golod homomorphism.*

Proof. The result follows from [8, Theorem 5.17] because of Example 2.9 and [18]. □

Lemma 6.3. *Let $(R, \mathfrak{M}, k)$ be a regular local ring, J be an ideal of R which is contained in $\mathfrak{M}^2$, and $a_1, \dots, a_m$ be an $R-$sequence which is contained in J. If the natural map $R/(a_1, \dots, a_m) \twoheadrightarrow R/J$ is Golod, then $a_1, \dots, a_m$ begins a minimal generating set for J.*

Proof. Let $A = R/J$ and $C = R/(\mathbf{a})$, where $\mathbf{a}$ represents $a_1, \dots, a_m$. If M is a module, then $\mu(M)$ is the minimal number of generators of M. Recall that

$$P_A(z) = \frac{(1+z)^{e_1}(1+z^3)^{e_3}(1+z^5)^{e_5}\cdots}{(1-z^2)^{e_2}(1-z^4)^{e_4}(1-z^6)^{e_6}\cdots},$$

where $e_1 = \dim R$ and $e_2 = \mu(J)$. (The deviations e_i were also considered at the beginning of section 4.) It follows that $P_A(z)P_R^{-1}(z) = (1 + \mu(J)z^2 + \dots)$. In a similar way, we see that $P_C(z)P_R^{-1}(z) = (1+mz^2+\dots)$. The map $C \twoheadrightarrow A$ is Golod; thus [8, (5.1)] ensures that

$$P_A(z) = P_C(z)\left(1 - z\left(P_C^A(z) - 1\right)\right)^{-1}. \tag{6.4}$$

The minimal resolution of A over C begins $\cdots \to C^\ell \to C \to A \to 0$, where $\ell = \mu(J/(\mathbf{a}))$. Multiply both sides of (6.4) by $P_R^{-1}(z)$ in order to obtain

$$\begin{aligned}(1 + \mu(J)z^2 + \cdots) = P_A(z)P_R^{-1}(z) &= (1 + mz^2 + \cdots)(1 + \ell z^2 + \cdots)\\ &= (1 + (m+\ell)z^2 + \cdots).\end{aligned}$$

It follows that $\mu(J) = m + \ell$; and the proof is complete. $\square$

Proposition 6.5. *Let $(R, \mathfrak{M}, k)$ be a regular local ring, $J \subseteq \mathfrak{M}^2$ be an ideal in R, $(A, \mathfrak{m}, k)$ be the quotient R/J, $T_\bullet = \mathrm{Tor}_\bullet^R(A, k)$, and $n = \dim_k T_1$. Suppose that $\dim_k T_1^2 = \binom{n}{2}$. If $\mathbf{a}$ is an $R-$sequence in J with the property that the natural map $C = R/(\mathbf{a}) \twoheadrightarrow A$ is a Golod map, then $T_2^2 \subseteq T_1T_3$.*

Proof. Fix a minimal generating set $x_1, \dots, x_e$ for $\mathfrak{M}$. Let $(\mathbb{K}, d)$ be the Koszul complex $R<X_1, \dots, X_e; d(X_i) = x_i>$ and let $\overline{\mathbb{K}} = A \otimes_R \mathbb{K}$. We view $T_\bullet$ as the homology of $\overline{\mathbb{K}}$. Eventually, we will prove that

$$Z_2(\overline{\mathbb{K}})Z_2(\overline{\mathbb{K}}) \subseteq Z_1(\overline{\mathbb{K}})Z_3(\overline{\mathbb{K}}) + B_4(\overline{\mathbb{K}}). \tag{6.6}$$

If $y \in \mathbb{K}$, then we write $\overline{y}$ to mean $1 \otimes y \in \overline{\mathbb{K}}$.

According to Lemma 6.3, we may select elements $y_1, \dots, y_n$ in $\mathbb{K}_1$ such that $d(y_1), \dots, d(y_m)$ is a minimal generating set for $(\mathbf{a})$ and $d(y_1), \dots, d(y_m), \dots, d(y_n)$ is a minimal generating set for J. We have chosen the elements y_i so that each $\overline{y}_i$ is in $Z_1(\overline{\mathbb{K}})$ and so that the corresponding classes $[\overline{y}_1], \dots, [\overline{y}_n]$ in homology form a basis for $H_1(\overline{\mathbb{K}})$. The hypothesis $\dim_k T_1^2 = \binom{n}{2}$ guarantees that the elements

$$[\overline{y}_i\overline{y}_j] \quad \text{such that } 1 \le i < j \le n \tag{6.7}$$

are linearly independent in $H_2(\overline{\mathbb{K}})$.

The ring C is a complete intersection; consequently,

$$(C \otimes_R \mathbb{K})<Y_1, \dots, Y_m; d(Y_i) = 1 \otimes y_i>$$

is the Tate resolution of k over C. The hypothesis $C \twoheadrightarrow A$ is Golod ensures that

$$\mathbb{L} = \overline{\mathbb{K}}<Y_1, \dots, Y_m; d(Y_i) = \overline{y}_i>

is a Golod algebra.

Let z_1 and z_2 be arbitrary elements of $Z_2(\overline{\mathbb{K}})$. The fact that $\mathbb{L}$ is a Golod algebra implies, among other things, that z_1z_2 is a boundary in $\mathbb{L}$; that is, there exists $\alpha \in \overline{\mathbb{K}}_5$, $\alpha_i \in \overline{\mathbb{K}}_3$ for $1 \le i \le m$, and $\alpha_{ij} \in \overline{\mathbb{K}}_1$ for $1 \le i \le j \le m$ such that

$$z_1z_2 = d\left(\alpha + \sum_{i=1}^{m}\alpha_i Y_i + \sum_{1\le i<j\le m}\alpha_{ij}Y_iY_j + \sum_{i=1}^{m}\alpha_{ii}Y_i^{(2)}\right).$$

When this equation is expanded, we obtain $\alpha_{ij} \in Z_1(\overline{\mathbb{K}})$ for $1 \le i \le j \le m$,

$$z_1z_2 = d(\alpha) - \sum_{i=1}^{m}\alpha_i\overline{y}_i, \quad \text{and} \tag{6.8}$$

$$d(\alpha_i) = \sum_{j=1}^{i-1}\alpha_{ji}\overline{y}_j + \alpha_{ii}\overline{y}_i + \sum_{j=i+1}^{m}\alpha_{ij}\overline{y}_j \quad \text{for } 1 \le i \le m. \tag{6.9}$$

A basis for $H_1(\overline{\mathbb{K}})$ has already been identified; thus, there exists $\beta_{ij} \in \overline{\mathbb{K}}_2$ and $a_{ijk} \in A$ for $1 \le i \le j \le m$ and $1 \le k \le n$, such that

$$\alpha_{ij} = \sum_{k=1}^{n} a_{ijk}\overline{y}_k + d(\beta_{ij}) \tag{6.10}$$

for $1 \le i \le j \le m$. If $1 \le i < j \le m$, then define $\alpha_{ji} = \alpha_{ij}$, $a_{jik} = a_{ijk}$ and $\beta_{ji} = \beta_{ij}$. Furthermore, if $m+1 \le i \le n$ or $m+1 \le j \le n$, then define $a_{jik} = a_{ijk} = 0$. It follows that (6.10) holds for $1 \le i, j \le m$ and (6.9) can be rewritten as

$$\begin{aligned} d(\alpha_i) &= \sum_{j=1}^{m}\alpha_{ij}\overline{y}_j = \sum_{j=1}^{m}\left(\sum_{k=1}^{n}a_{ijk}\overline{y}_k + d(\beta_{ij})\right)\overline{y}_j \\ &= \sum_{1\le k<j\le n}(a_{ijk} - a_{ikj})\overline{y}_k\overline{y}_j + d\left(\sum_{j=1}^{m}\beta_{ij}\overline{y}_j\right). \end{aligned} \tag{6.11}$$

Use (6.7) to see that $a_{ijk} - a_{ikj} \in \mathfrak{m}$ for $1 \le i, j, k \le n$. It is not difficult to find $\gamma_{ijk} \in \overline{\mathbb{K}}_1$ such that

$$d(\gamma_{ijk}) = a_{ijk} - a_{ikj}, \quad \gamma_{ijk} + \gamma_{ikj} = 0, \quad \gamma_{ijk} + \gamma_{jki} + \gamma_{kij} = 0 \quad \text{for } 1 \le i, j, k \le n, \text{ and}$$
$$\gamma_{ijk} = 0 \quad \text{for } m+1 \le i \le n \text{ and } 1 \le j, k \le n.$$

(For example, if $1 \le i < j < k \le n$, then select γ_{ijk} and γ_{jki} with $d(\gamma_{ijk}) = a_{ijk} - a_{ikj}$ and $d(\gamma_{jki}) = a_{jki} - a_{jik}$. Define $\gamma_{kij} = -\gamma_{ijk} - \gamma_{jki}$, $\gamma_{ikj} = -\gamma_{ijk}$, and

$\gamma_{jik} = -\gamma_{jki}$, $\gamma_{kji} = -\gamma_{kij}$. This procedure must be modified slightly if there are repetitions among the indices i, j, k.) It now follows from (6.11) that

$$d(\alpha_i) = d\left(\sum_{1\le k<j\le n} \gamma_{ijk}\overline{y}_k\overline{y}_j + \sum_{j=1}^{m} \beta_{ij}\overline{y}_j\right);$$

thus, for $1 \le i \le m$, there exists $w_i \in Z_3(\overline{\mathbb{K}})$ such that

$$\alpha_i = \sum_{1\le k<j\le n} \gamma_{ijk}\overline{y}_k\overline{y}_j + \sum_{j=1}^{m} \beta_{ij}\overline{y}_j + w_i. \tag{6.12}$$

When (6.12) is combined with (6.8), we obtain

$$z_1z_2 = d(\alpha) - \sum_{i=1}^{m} w_i\overline{y}_i.$$

Line (6.6) has been established; and the proof is complete. □

References

1. D. Anick, *A counterexample to a conjecture of Serre*, Annals of Math. **115** (1982), 1–33.
2. L. Avramov, *Small homomorphisms of local rings*, J. Alg. **50** (1978), 400–453.
3. L. Avramov, *Golod homomorphisms*, Algebra, Algebraic Topology and their Interactions, Lecture Notes in Mathematics **1183**, Springer Verlag, Berlin Heidelberg New York, 1986, pp. 59–78.
4. L. Avramov, *Homological asymptotics of modules over local rings*, Commutative Algebra, Mathematical Sciences Research Institute Publications **15**, Springer Verlag, Berlin Heidelberg New York, 1989, pp. 33–62.
5. L. Avramov, *Modules of finite virtual projective dimension*, Invent. math. **96** (1989), 71–101.
6. L. Avramov, *Local rings over which all modules have rational Poincaré series*, J. Pure Appl. Algebra **91** (1994), 29–48.
7. L. Avramov and Y. Félix, *Espaces de Golod*, Astérisque **191** (1990), 29–34.
8. L. Avramov, A. Kustin, and M. Miller, *Poincaré series of modules over local rings of small embedding codepth or small linking number*, J. Alg. **118** (1988), 162–204.
9. D. Buchsbaum and D. Eisenbud, *Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension 3*, Amer. J. Math. **99** (1977), 447–485.
10. D. Eisenbud, *Homological algebra on a complete intersection, with an application to group representations*, Trans. Amer. Math. Soc. **260** (1980), 35–64.
11. V. Gasharov and I. Peeva, *Boundedness versus periodicity over commutative local rings*, Trans. Amer. Math. Soc. **320** (1990), 569–580.
12. A. Grothendieck, *Eléments de Géométrie Algébrique III*, IHES Publications Math. No. 11, Paris, 1961.
13. T. Gulliksen and G. Levin, *Homology of local rings*, Queen's papers in pure and applied mathematics **20**, Queen's University, Kingston, Ontario, 1969.
14. J. Herzog, *Homological properties of the module of differentials*, Atas da 6^a escola de álgebra Recife, Coleção Atas. Soc. Brasileria de Mat. **14**, Rio de Janeiro, 1981, pp. 33–64.
15. C. Jacobsson, A. Kustin, and M. Miller, *The Poincaré series of a codimension four Gorenstein ring is rational*, J. Pure Appl. Algebra **38** (1985), 255–275.
16. T. Józefiak, *Tate resolutions for commutative graded algebras over a local ring*, Fund. Math. **74** (1972), 209-231.

17. A. Kustin, *Classification of the Tor-algebras of codimension four almost complete intersections*, Trans. Amer. Math. Soc. **339** (1993), 61–85.
18. A. Kustin, *The minimal resolution of a codimension four almost complete intersection is a DG-algebra*, J. Alg. **168** (1994), 271–295.
19. S. Palmer, *Algebra structures on resolutions of rings defined by grade four almost complete intersections*, Ph.D. thesis, University of South Carolina, 1990.
20. S. Palmer, *Algebra structures on resolutions of rings defined by grade four almost complete intersections*, J. Alg. **159** (1993), 1–46.

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