COMPLEXES WHICH ARISE FROM A MATRIX AND A VECTOR: RESOLUTIONS OF DIVISORS ON CERTAIN VARIETIES OF COMPLEXES

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Abstract. Consider the polynomial ring $R = R_0[X,Y]$ where R_0 is a normal domain, and $X_{1\times g}$ and $Y_{g\times f}$ are matrices of indeterminates. The $R-$ideal $J = I_1(X)+I_{\min\{f,g\}}(Y)$ defines a variety of complexes over R_0. The divisor class group of R/J is isomorphic to $\mathrm{C}\ell\,(R_0)\oplus\mathbb{Z}[I']$, where I' is an ideal of R/J generated by appropriately chosen lower order minors of Y. We produce the minimal $R-$free resolution of $i[I']$ for all integers $i\geq -1$. If f is greater than or equal to g, then J is a generic residual intersection of the generic grade g complete intersection $I_1(X)$. The resolutions that we produce in this case are, in many ways, analogous to resolutions of divisors on generic residual intersections of grade two perfect ideals or grade three Gorenstein ideals.

All rings in this paper are commutative noetherian rings with one; f and g always represent non-negative integers. Let R_1 be a ring. Consider linear forms $z_1,\dots,z_f$ in the polynomial ring $R=R_1[x_1,\dots,x_g]$, where $z_j=\sum_{i=1}^{g} y_{ij}x_i$. If S is the quotient ring $R/(z_1,\dots,z_f)$, then $\operatorname{Spec} S\to\operatorname{Spec} R_1$ represents a family of linear subspaces of codimension at most $\min\{f,g\}$ in affine $g-$space. Avramov [4] refers to the set of points in $\operatorname{Spec} R_1$ over which the fibre is a linear subspace of codimension strictly less than $\min\{f,g\}$ as the "discriminantal locus" of the family $\operatorname{Spec} S\to\operatorname{Spec} R_1$. The inverse image in $\operatorname{Spec} S$ of this discriminantal locus is defined by the ideal $J=(z_1,\dots,z_f)+I_{\min\{f,\,g\}}(Y)$ of R (where "$I_t(Y)$" denotes the ideal generated by the $t\times t$ minors of the matrix $Y=(y_{ij})$). In this paper, we produce a family of complexes of free $R-$modules associated to the quotient ring R/J. In particular, in the generic case we resolve half of the divisor class group of R/J.

Suppose that $z_1,\dots,z_f$ are homogeneous polynomials in $\mathbb{C}[x_1,\dots,x_g]$. Since

$$\sum_{i=1}^{g}\frac{\partial z_j}{\partial x_i}x_i=(\deg z_j)z_j,$$

it is observed in [11] that an alternate significance may be attached to the ideal J. If $f<g$ and $Y=(y_{ij})$ is the matrix with $y_{ij}=\frac{1}{(\deg z_j)}\frac{\partial z_j}{\partial x_i}$, then $J=(z_1,\dots,z_f)+$

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$I_f(Y)$ defines the singular locus of the complete intersection defined by $(z_1, \dots, z_f)$ in projective $(g-1)-$space.

Let X and Y be matrices, of shape $1\times g$ and $g\times f$, respectively, over the arbitrary commutative noetherian ring R. We consider the ideal J of R which is defined by

$$J = I_1(XY) + I_{\min\{f,\,g\}}(Y). \tag{0.1}$$

If J is a proper ideal of R, then it is known (see, for example, [11, Theorem 5.1] and [8, Theorem 3.6]) that

$$\operatorname{grade}(J) \le \max\{f, g\}; \tag{0.2}$$

furthermore, if equality holds in (0.2), then J is a perfect ideal. In this paper we are concerned with ideals J of the form of (0.1) for which equality holds in (0.2). The history of such ideals is quite rich. We begin with their homological history. Northcott [27] found the $R-$resolution of R/J in the case that $f = g$. In this case, J is called a *Northcott ideal*; it is an almost complete intersection, and it is one link from a complete intersection. Herzog [13] resolved R/J when $f = g-1$. In this case, J is called a *Herzog ideal*; furthermore, the ideal is Gorenstein and it is two links from a complete intersection. It is essentially the case (see [6, Section 3]) that every ideal which is a small number of links from a complete intersection is a Northcott ideal or a Herzog ideal. (See [5, Theorem 1.5] for up-to-date homological information about ideals of small linking number.) Buchsbaum and Eisenbud [11, Section 5] resolved R/J whenever $f \le g$. (Avramov [3] also has resolved R/J when $f \le g$. His resolution is not minimal; nonetheless, he is able to use it in order to compute the Poincaré series

$$P_{R/J}(t) = \sum_{n=0}^{\infty} \dim_k \operatorname{Tor}_n^{R/J}(k,k)\, t^n$$

of R/J in terms of the Poincaré series of R whenever $(R, \mathfrak{m}, k)$ is a local ring and $f = 2$.) If $g \le f$, then R/J is resolved in [8]. In this case, Huneke and Ulrich [19] have shown that J is an $f-$residual intersection of a complete intersection (provided X, Y, and R are sufficiently generic; for example, the statement holds if $I_1(X)$ is a complete intersection and R is a Gorenstein ring.)

We now turn to the divisorial history of ideals of the form of J. In this discussion the entries of X and Y are indeterminates over a normal domain R_0, and R is the polynomial ring $R_0[X, Y]$. (We mean, of course, that $R = R_0[\{x_i\}, \{y_{ij}\}]$.) In [4] Avramov considered the quotient R/J, with $f \le g$, as the symmetric algebra $S_\bullet^{(R_1/K)}(M/KM)$, where R_1 is the polynomial ring $R_0[Y]$, K is the ideal $I_f(Y)$ of R_1, and M is the R_1-module presented by

$$R_1^f \xrightarrow{Y} R_1^g \to M \to 0.$$

He produced an R_1-resolution of each symmetric power $S_t(M/KM)$, he proved that R/J is a normal domain, and he proved that the inclusion map $R_0 \to R/J$ induces an isomorphism

$$\mathrm{C}\ell\,(R/J) \cong \mathrm{C}\ell\,(R_0) \oplus \mathbb{Z} \tag{0.3}$$

whenever $2 \le f \le g$.

More detailed information about the divisor class group of R/J can be obtained once we consider R/J from the point of view of the variety of complexes. In affine space $\mathbb{A}^{fg+g}(R_0)$, consider the variety V of all complexes

$$0 \to R_0^f \xrightarrow{\theta_2} R_0^g \xrightarrow{\theta_1} R_0$$

with $\operatorname{rank}\theta_2 < \min\{f, g\}$. DeConcini and Strickland [12] have used Hodge algebra techniques to prove that J is the defining ideal of the variety V and that J is a perfect ideal with grade equal to the maximum of $\{f, g\}$. Furthermore, they proved that R/J is a normal domain. Bruns [7] and Yoshino [32] extended the work of [12] by calculating the divisor class groups of the rings in question. In particular, they proved that (0.3) holds for all f and g with $2 \le \min\{f, g\}$. Furthermore, Bruns has identified a generator $[I']$ for the summand $\mathbb{Z}$ in (0.3). Let I' be the ideal

$$\frac{I_{\min\{f,g\}-1}(Y') + J}{J} \tag{0.4}$$

of R/J, where Y' is the

$$\begin{cases} (f-1)\times f \text{ submatrix of } Y \text{ consisting of rows} & 1 \text{ to } f-1, \quad \text{if } f<g \\ g\times(g-1) \text{ submatrix of Y consisting of columns } 1 \text{ to } g-1, & \text{if } g \le f. \end{cases}$$

Bruns has proved that I' is a divisorial ideal of R/J and that the class $[I']$ of I' in $C\ell\,(R/J)$ generates the summand $\mathbb{Z}$ in (0.3).

We are interested in resolving $R-$modules which represent elements of the class group of R/J. We observed above that R/J (which represents the class $i[I']$ for $i = 0$) has been resolved for all f and g. If $g \le f$, then Pellikaan ([28] or [29]) has resolved the representative $I(X)/I(XY)$ of the class $i[I']$ for $i = 1$. In Theorem 9.2 we give the minimal $R-$resolution of a representative of $i[I']$ for all $i \ge -1$ and for all f and g with $2 \le \min\{f, g\}$.

The impetus for resolving elements of the divisor class group of R/J is an ongoing project to resolve residual intersections. (See [2], [17], or [19] for an introduction to residual intersections; see [8], [25], and [26] for the complete statement of the progress that has been made to date on this project.) The progress that has been made can be summarized as: "The best chance for understanding the residual intersection of an ideal I occurs when I is in the $\underline{\text{l}}$inkage $\underline{\text{c}}$lass of a $\underline{\text{c}}$omplete $\underline{\text{i}}$ntersection (licci). The most studied examples of licci ideals are complete intersections of arbitrary grade, grade two perfect ideals, and grade three Gorenstein ideals. If I is an ideal from the above list, then all $f-$residual intersections of I have been resolved. If I is a grade two perfect ideal or if I is a grade three Gorenstein ideal, then an exciting family of complexes ($\{\mathfrak{C}^i\}$, $\{\mathfrak{D}^i\}$, respectively) has been associated to I." The preceding interpretation of history naturally leads one to formulate the following project.

Goal 0.5. Find a family of complexes $\{\mathfrak{B}^i\}$ which is associated to a complete intersection and which is analogous to the families $\{\mathfrak{C}^i\}$ and $\{\mathfrak{D}^i\}$.

We recall the properties of the families $\{\mathfrak{C}^i\}$ and $\{\mathfrak{D}^i\}$. In this discussion I is a grade two perfect ideal or a grade three Gorenstein ideal and

$$R^n \xrightarrow{P} R^g \xrightarrow{X} R \to (R/I) \to 0$$

induces a minimal presentation of I. If I is a grade two perfect ideal, then $n = g-1$, P is a $g \times n$ generic matrix, and the entries of X are maximal order minors of P. If I is a grade three Gorenstein ideal, then $n = g$, P is a generic $g \times n$ alternating matrix, and the entries of X are maximal order pfaffians of P. In order to simplify the exposition we assume that the ring R is the polynomial ring $k[P, Y]$ where k is a field and Y is a $g \times f$ generic matrix. Given this data with $f \ge \operatorname{grade} I$, let J be the $f-$residual intersection $(I_1(XY) : I)$, ρ be the map

$$\rho = [P\, Y] \colon E = R^n \oplus R^f \to G = R^g,$$

and N be the integer

$$N = f + 1 - \operatorname{grade} I.$$

For each integer i, one can form the complex $\mathfrak{C}^i = \mathfrak{C}^i(\rho)$ (if I is a grade two perfect ideal) or $\mathfrak{D}^i = \mathfrak{D}^i(\rho)$ (if I is a grade three Gorenstein ideal). The families $\{\mathfrak{C}^i\}$ and $\{\mathfrak{D}^i\}$ satisfy the following properties.

(0.6) The complexes $\mathfrak{C}^0$ and $\mathfrak{D}^0$ each resolve R/J.

(0.7) The divisor class group of R/J is the infinite cyclic group $\mathbb{Z}[\operatorname{coker}(\rho)]$.

(0.8) If $i \ge -1$, then $\mathfrak{C}^i$ and $\mathfrak{D}^i$ each resolve a representative of the class $i[\operatorname{coker}(\rho)]$ from $\mathrm{C}\ell\,(R/J)$.

(0.9) The canonical class in the $\mathrm{C}\ell\,(R/J)$ is equal to $N[\operatorname{coker}(\rho)]$.

(0.10) The complexes $\{\mathfrak{C}^i\}$ and $\{\mathfrak{D}^i\}$ satisfy

$$\mathfrak{C}^i \cong \left(\mathfrak{C}^{N-i}\right)^* [-s] \qquad \text{and} \qquad \mathfrak{D}^i \cong \left(\mathfrak{D}^{N-i}\right)^* [-s]$$

for $s = f$.

(0.11) If M is a reflexive $(R/J)-$module of rank one and $[M] = i[\operatorname{coker}(\rho)]$ in $\mathrm{C}\ell\,(R/J)$ for some integer i, then M is a Cohen-Macaulay module if and only if $-1 \le i \le N+1$.

(0.12) If $\widetilde{\rho} = [P\, \widetilde{Y}]$, where $\widetilde{Y}$ is the submatrix of Y which consists of columns 1 to $f-1$, then, for each integer i, there is a short exact sequence of complexes

$$0 \to \mathfrak{C}^i(\widetilde{\rho}) \to \mathfrak{C}^i \to \mathfrak{C}^{i-1}(\widetilde{\rho})[-1] \to 0 \quad \text{or} \quad 0 \to \mathfrak{D}^i(\widetilde{\rho}) \to \mathfrak{D}^i \to \mathfrak{D}^{i-1}(\widetilde{\rho})[-1] \to 0.$$

In the present paper we need to know the properties of the complexes $\mathfrak{C}^i$ and $\mathfrak{D}^i$, but not their exact description; however, a rough description can be given quickly. The complex $\mathfrak{C}^i$, although well-known, does not seem to have a name. It is

$$\cdots \to D_2G^* \otimes \bigwedge^{g+i+2} E \to D_1G^* \otimes \bigwedge^{g+i+1} E \to D_0G^* \otimes \bigwedge^{g+i} E \to S_0G \otimes \bigwedge^{i} E$$
$$\to S_1G \otimes \bigwedge^{i-1} E \to S_2G \otimes \bigwedge^{i-2} E \to \dots$$

with $S_0G \otimes \bigwedge^i E$ in position i. (It appears that Buchsbaum and Eisenbud [10] or Kirby [23] first considered the full family of complexes $\{\mathfrak{C}^i\}$. See [26, section 2] for more details and history.) The family $\{\mathfrak{D}^i\}$ is defined in [26]. Roughly speaking, the complex $\mathfrak{D}^i$ is obtained by pasting a graded strand of the algebra $(S_\bullet G \otimes \bigwedge^\bullet E)/\mathcal{I}$ together with the dual of a different graded strand of $(S_\bullet G \otimes \bigwedge^\bullet E)/\mathcal{I}$. (In this discussion $S_\bullet G \otimes \bigwedge^\bullet E$ is the Koszul algebra associated to the map ρ, and $\mathcal{I}$ is a two generated of $S_\bullet G \otimes \bigwedge^\bullet E$.) The position in $\mathfrak{D}^i$ where the two strands are patched together involves pfaffians, of various sizes, of the alternating map which corresponds to almost alternating map ρ.

At any rate, Goal 0.5 is completely accomplished in this paper. The complexes that we define are best described if the data is given in terms of maps (rather than matrices). Throughout this paper F and G are free $R-$modules of rank f and g, respectively, and

$$F \xrightarrow{\Upsilon} G \xrightarrow{\Xi} R \tag{0.13}$$

are $R-$module homomorphisms. Given this data, we consider two families of complexes: $\{\mathfrak{B}^i\}$ and $\{\mathfrak{b}^i\}$. The complexes $\{\mathfrak{B}^i\}$ accomplish Goal 0.5. The complete intersection $I = I_1(\Xi)$ is presented by

$$\bigwedge^2 G \xrightarrow{\bigwedge^2 \Xi} G \xrightarrow{\Xi} R \to \frac{R}{I} \to 0,$$

and the role of the map $\rho : E \to G$ is played by

$$\rho = [\bigwedge^2 \Xi \quad \Upsilon] \colon \bigwedge^2 G \oplus F \to G.$$

The complexes $\mathfrak{B}^i$ are defined in section 2. The properties of these complexes are established in sections 3 and 4. A summary of the properties of the complexes $\mathfrak{B}^i$ (in the generic case) is contained in section 9. The non-generic case is treated in section 10. In particular, property (0.12) is proved in Proposition 3.13 and properties (0.6) – (0.11) for $2 \le g \le f$ may all be found in Theorem 9.2. It is worth noting that our treatment of the $\mathfrak{B}^i$ is relatively painless. (In particular, we have separated our discussion of the $\mathfrak{B}^i$ from our discussion of the more complicated complexes $\mathfrak{b}^i$.) For example, the proof of the acyclicity of the $\mathfrak{B}^i$ is contained in section 4, which is quite short. The proof proceeds by induction on f. Property (0.12) shows that the complex $\mathfrak{B}^i$ is the mapping cone of two complexes which are built from a smaller f. The induction hypothesis yields that most of the homology of $\mathfrak{B}^i$ is trivial. The induction begins with $f = 0$. This situation has no meaning in terms of either varieties of complexes or residual intersections; nonetheless, the complex $\mathfrak{B}^i$ is defined when $f = 0$ and it is the Buchsbaum-Eisenbud resolution $\mathbf{L}_i^1(\Xi)$ of the i^{th} power of an ideal generated by a regular sequence. See Observation 3.10.

The expression "J is an $f-$residual intersection of I" has been given meaning only when $f \ge \operatorname{grade} I$. On the other hand, the complexes $\mathfrak{B}^i$, $\mathfrak{C}^i$, and $\mathfrak{D}^i$ are defined for all $f \ge 0$. Furthermore, if (0.6) is used as a definition of J when $0 \le f \le$

$(\operatorname{grade} I)-1$, then many of the other properties hold. In particular, Propositions 9.6 and 3.13 show that properties (0.7) – (0.12) all hold when $f=(\operatorname{grade} I)-1$. Huneke and Ulrich [20] provide some explanation why properties (0.7), (0.9), and (0.11) hold in the case of residual intersection (i.e., $f \geq \operatorname{grade} I$); however, a complete understanding of which types of ideals give rise to a family of complexes analogous to $\{\mathfrak{B}^i\}$, $\{\mathfrak{C}^i\}$, and $\{\mathfrak{D}^i\}$ is not yet available.

The second major impetus for this paper turns out to have been a false hope. The Buchsbaum-Eisenbud resolution of R/J for $f \leq g$ (where J is defined in (0.1)) bears some faint resemblance to the Bruns-Kustin-Miller resolution of R/J for $g \leq f$. It seemed reasonable to look for one big family of complexes which specialized to give the Buchsbaum-Eisenbud resolution (and resolutions of divisors in this case) as well as the Bruns-Kustin-Miller resolution (and resolutions of divisors in this case.) This search led to the complex $\mathfrak{b}^i$ which resolves the divisor $i[I']$ when $f<g$ (in the notation of (0.4)). However, now that we have the family $\{\mathfrak{b}^i\}$ it is impossible to imagine **one** family of complexes which contains $\mathfrak{B}^i$ when $g \leq f$ and also contains $\mathfrak{b}^i$ when $f<g$. (In particular, most of the maps in $\mathfrak{B}^i$ are linear; but half the maps in $\mathfrak{b}^i$ are quadratic. See Figure 3.4 and Figure 6.2.) It is curious, however, that, in some sense, the family $\{\mathfrak{b}^i\}$ satisfies properties which are similar to (0.6) – (0.12). In the case of the $\mathfrak{b}^i$ there is no ideal I, there is no presentation map P of I, and there is no interpretation in terms of residual intersection. Nonetheless, if N is set equal to $g-f-1$, s is set equal to g, and ρ is set equal to the map

$$(0.14)\quad \rho=[1\otimes \Upsilon^*\Xi^*(1) \quad \Upsilon^*\otimes 1]\colon (F^*\otimes \textstyle\bigwedge^{f-1} F^*)\oplus(G^*\otimes \textstyle\bigwedge^{f} F^*)\to F^*\otimes \textstyle\bigwedge^{f} F^*,$$

then, in Theorem 9.2, we prove that properties (0.6) – (0.11) hold for the family $\{\mathfrak{b}^i\}$ in the generic case whenever $g>f>1$. A version of (0.12) which holds for the $\mathfrak{b}^i$ is established in Theorem 6.10.

A review of multilinear algebra is given in section 1. The complexes $\{\mathfrak{B}^i\}$ and $\{\mathfrak{b}^i\}$ are defined in sections 2 and 5, respectively. Elementary properties of the $\mathfrak{B}^i$ and the $\mathfrak{b}^i$ are recorded in sections 3 and 6, respectively. We establish the acyclicity of the $\mathfrak{B}^i$ and the $\mathfrak{b}^i$ (for the generic case) in sections 4 and 7, respectively. Section 4 is much shorter than section 7 mainly because the "base case" ($f=0$) is already well known for the $\mathfrak{B}^i$; but, the "base case" ($f=g$) is brand new for the $\mathfrak{b}^i$. One of the keys to the proof in section 7 is Theorem 7.22, which is a result about linkage of Huneke-Ulrich deviation two Gorenstein ideals. Section 8 is concerned with the $\mathfrak{b}^i$ in the "degenerate situation" $g \leq f-1$. It turns out that the $\mathfrak{b}^i$ are acyclic if $g=f-1$; but, that these complexes have nontrivial homology when $g \leq f-2$. Section 9 summarizes what is known in the generic case. In section 10 we give conditions under which specialization preserves the acyclicity of the complexes $\mathfrak{B}^i$ and $\mathfrak{b}^i$, and we interpret the complexes $\{\mathfrak{B}^i\}$ in the context of residual intersection.

The reader should consult [26] or [9] for any definitions or conventions that we have neglected to record.

Section 1. Multilinear algebra and other preliminary concepts.

Fix a commutative noetherian ring R. All $R-$modules that we consider are finitely generated. If M is an $R-$module, then $M^*=\operatorname{Hom}_R(M,R)$ is the dual of

M. Let E be a free $R-$module of rank e. We do not distinguish between E and its double dual E^{**}. The symmetric algebra $S_\bullet E$, the exterior algebra $\bigwedge^\bullet E$, and the divided power algebra $D_\bullet E$ appear throughout the rest of the paper. (The formal properties of these algebras and an expanded version of the following discussion may be found in [11] and [26].) Each of these algebras A comes equipped with a multiplication $\mu\colon A\otimes A\to A$, and a co-multiplication $\varDelta\colon A\to A\otimes A$. (The symbol $\otimes$ always means tensor over R.) For each non-negative integer i there are **canonical** perfect pairings

$$\ll,\gg\colon \bigwedge^i E^*\otimes\bigwedge^i E\to R \qquad\text{and}\qquad \ll,\gg\colon S_iE^*\otimes D_iE\to R, \tag{1.1}$$

which are induced by co-multiplication followed by the evaluation map

$$<,>\colon E^*\otimes E\to R.$$

Every free module that we consider is oriented. In other words, there are **fixed** isomorphisms

$$[\quad]:\bigwedge^e E\to R \qquad\text{and}\qquad [\quad]:\bigwedge^e E^*\to R$$

which are compatible with the perfect pairing of (1.1) in the sense that they satisfy

$$\ll x,y\gg=[x][y]$$

for all $x\in\bigwedge^e E^*$ and $y\in\bigwedge^e E$. Exterior multiplication followed by the orientation isomorphism produces a further **canonical** perfect pairing

$$\ll,\gg\colon \bigwedge^i E\otimes\bigwedge^{e-i} E\to R. \tag{1.2}$$

If $\rho\colon E\to G$ is a map of free $R-$modules, then we refer to the differential algebras

$$(S_\bullet G\otimes\bigwedge^\bullet E\,,\,\partial_\rho) \qquad\text{and}\qquad (D_\bullet G^*\otimes\bigwedge^\bullet E\,,\,\delta_\rho)$$

as the *Koszul algebra* and the *Eagon-Northcott algebra* associated to ρ, respectively. (Often we will write ∂ and δ in place of ∂_ρ and δ_ρ.) The differentials

$$\partial\colon S_aG\otimes\bigwedge^b E\to S_{a+1}G\otimes\bigwedge^{b-1}E \qquad\text{and}\qquad \delta\colon D_aG^*\otimes\bigwedge^b E\to D_{a-1}G^*\otimes\bigwedge^{b-1}E$$

are well-known, but may also be found in section one of [26]. Recall that the perfect pairings of (1.1) and (1.2) induce the perfect pairing

$$\ll,\gg\colon \left(S_aG\otimes\bigwedge^b E\right)\otimes\left(D_aG^*\otimes\bigwedge^{e-b}E\right)\to R, \tag{1.3}$$

which satisfies

$$\ll \partial(x), y \gg = (-1)^{b-1} \ll x, \delta(y) \gg \tag{1.4}$$

for all $x \in (S_aG \otimes \bigwedge^b E)$ and $y \in (D_{a+1}G^* \otimes \bigwedge^{e-b+1} E)$. It follows that (1.3) induces an isomorphism of complexes:

(1.5)

$$\begin{array}{ccccccc} \dots \longrightarrow & S_aG \otimes \bigwedge^b E & \xrightarrow{\partial} & S_{a+1}G \otimes \bigwedge^{b-1} E & \longrightarrow \dots \\ & \downarrow & & \downarrow & \\ \dots \longrightarrow & \left(D_aG^* \otimes \bigwedge^{e-b} E\right)^* & \xrightarrow{\delta^*} & \left(D_{a+1}G^* \otimes \bigwedge^{e-b+1} E\right)^* & \longrightarrow \dots & . \end{array}$$

We are particularly interested in studying the Koszul algebra and the Eagon-Northcott algebra which are associated to the identity map on E^*. It is well-known that almost all graded strands of these algebras are split exact. (The only exceptions are

$$0 \xrightarrow{\partial} S_0E^* \otimes \bigwedge^0 E^* \xrightarrow{\partial} 0 \quad \text{and} \quad 0 \xrightarrow{\delta} D_0E \otimes \bigwedge^e E^* \xrightarrow{\delta} 0.$$

Notice that we always consider S_iE, $\bigwedge^i E$, and D_iE to be zero whenever i is negative.) We now establish the existence of certain homotopies on these algebras which have particularly nice properties. These homotopies are used in section 5 when we construct the complex $\mathfrak{b}^i$.

Proposition 1.6. *Let E be a free $R-$module of rank e, and let ∂ and δ represent the differentials from the Koszul algebra and the Eagon-Northcott algebra associated to the identity map on E^*. Then there exist a family of maps*

$$\mathbf{s}\colon S_aE^* \otimes \bigwedge^b E^* \to S_{a-1}E^* \otimes \bigwedge^{b+1} E^* \quad \textit{and} \quad \mathbf{t}\colon D_aE \otimes \bigwedge^b E^* \to D_{a+1}E \otimes \bigwedge^{b+1} E^*$$

which has the following properties:

(a) $\mathbf{s}\,\partial + \partial\,\mathbf{s}$ *is the identity on* $S_aE^* \otimes \bigwedge^b E^*$ *for all integers a and b provided* $(a, b) \neq (0, 0)$,
(b) $\mathbf{s}x\mathbf{s} = 0$ *for any* $x \in S_0E^* \otimes \bigwedge^\bullet E^*$,
(c) $\mathbf{t}\,\delta + \delta\,\mathbf{t}$ *is the identity on* $D_aE \otimes \bigwedge^b E^*$ *for all integers a and b provided* $(a, b) \neq (0, e)$,
(d) $\mathbf{t}x\mathbf{t} = 0$ *for any* $x \in D_0E \otimes \bigwedge^\bullet E^*$, *and*
(e) *if* $\ll, \gg$ *is the perfect pairing of (1.3), then* $\ll \mathbf{s}(x), y \gg = (-1)^b \ll x, \mathbf{t}(y) \gg$ *for all* $x \in \left(S_aE^* \otimes \bigwedge^b E^*\right)$ *and* $y \in \left(D_{a-1}E \otimes \bigwedge^{e-b-1} E^*\right)$.

Proof. As soon as we have found a homotopy $\mathbf{s}$ which satisfies (a) and (b), we may take (e) to be the definition of $\mathbf{t}$. It is clear from (1.4) that $\mathbf{t}$ will then satisfy (c) and (d).

We now define $\mathbf{s}$. Let $\varepsilon_1, \dots, \varepsilon_e$ be a basis for E^*. Consider the basis element

$$x = \varepsilon_{i_1} \dots \varepsilon_{i_a} \otimes \varepsilon_{j_1} \wedge \dots \wedge \varepsilon_{j_b} \in S_aE^* \otimes \bigwedge^b E^*$$

where $1 \le i_1 \le \dots \le i_a \le e$ and $1 \le j_1 < \dots < j_b \le e$. Define

$$\mathbf{s}(x) = \begin{cases} 0, & \text{if} \quad i_a \le j_b, \quad \text{or} \quad a = 0 \\ \varepsilon_{i_1} \dots \varepsilon_{i_{a-1}} \otimes \varepsilon_{i_a} \wedge \varepsilon_{j_1} \wedge \dots \wedge \varepsilon_{j_b}, & \text{if} \quad j_b < i_a, \quad \text{or} \quad 0 = b < a. \end{cases}$$

It is apparent that (b) holds. To show (a) we consider two cases. If $i_a \le j_b$ or $0 = a < b$, then $\partial\, \mathbf{s}(x) = 0$ and $\mathbf{s}\, \partial(x) = x$. If $j_b < i_a$ or $0 = b < a$, then

$$\partial\, \mathbf{s}(x) = x + \sum_{k=1}^{b} (-1)^k \varepsilon_{i_1} \dots \varepsilon_{i_{a-1}} \varepsilon_{j_k} \otimes \varepsilon_{i_a} \wedge \varepsilon_{j_1} \wedge \dots \wedge \widehat{\varepsilon_{j_k}} \wedge \dots \wedge \varepsilon_{j_b}, \quad \text{and}$$

$$\mathbf{s}\, \partial(x) = \sum_{k=1}^{b} (-1)^{k+1} \varepsilon_{i_1} \dots \varepsilon_{i_{a-1}} \varepsilon_{j_k} \otimes \varepsilon_{i_a} \wedge \varepsilon_{j_1} \wedge \dots \wedge \widehat{\varepsilon_{j_k}} \wedge \dots \wedge \varepsilon_{j_b}. \quad \square$$

The building blocks of the complexes $\mathfrak{B}^i$ and $\mathfrak{b}^i$ are Schur functors. We find the following notation to be most convenient.

Definition 1.7. *Let E be a free $R-$module. If a and b are integers, then*

(a) $\mathfrak{L}_a^b E = \operatorname{Ker}(\partial : S_a E \otimes \bigwedge^b E \to S_{a+1} E \otimes \bigwedge^{b-1} E)$, *and*
(b) $K_a^b E = \operatorname{Ker}(\delta : D_a E \otimes \bigwedge^b E^* \to D_{a-1} E \otimes \bigwedge^{b-1} E^*)$,

where $(S_\bullet E \otimes \bigwedge^\bullet E, \partial)$ is the Koszul algebra associated to the identity map on E, and $(D_\bullet E \otimes \bigwedge^\bullet E^*, \delta)$ is the Eagon-Northcott algebra associated to the identity map on E^*.

It follows from Proposition 1.6 that

$$\mathfrak{L}_0^b E = \begin{cases} R, & \text{if } b = 0 \\ 0, & \text{otherwise,} \end{cases} \qquad \text{and} \qquad K_a^e E = \begin{cases} R, & \text{if } a = 0 \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, we also see that

$$S_{a-2} E \otimes \bigwedge^{b+2} E \xrightarrow{\partial} S_{a-1} E \otimes \bigwedge^{b+1} E \xrightarrow{\partial} \mathfrak{L}_a^b E \to 0$$

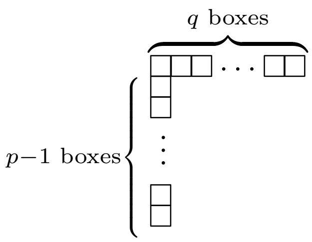
is exact provided $(a, b) \ne (0, 0)$ and $(a, b) \ne (1, -1)$, and

$$D_{a+2} E \otimes \bigwedge^{b+2} E^* \xrightarrow{\delta} D_{a+1} E \otimes \bigwedge^{b+1} E^* \xrightarrow{\delta} K_a^b E \to 0 \tag{1.8}$$

is exact provided $(a, b) \ne (0, e)$ and $(a, b) \ne (-1, e-1)$. The module $\mathfrak{L}_a^b E$ of Definition 1.7 is exactly the same as the the Schur functor $L_a^{b+1} E$ which is defined on page 260 of [11]. In more modern terminology, for example [1, Definition II.1.3], if p and q are positive integers, then $L_p^q E$ is the Schur functor $L_\lambda E$ of E with respect to the partition

$$\lambda = (q, \underbrace{1, \dots, 1}_{p-1 \text{ times}}).$$

The partition λ is represented by the hook



.

The module K_a^bE of Definition 1.7 is exactly the same as the module K_a^bE of [8]; furthermore, a quick comparison of (1.8) and (1.5) yields that

$$(K_a^bE)^* \cong \mathfrak{L}_{a+1}^{e-b-1}E^* \tag{1.9}$$

for all integers a and b provided $(a,b) \neq (0,e)$ and $(a,b) \neq (-1,e-1)$. At any rate, K_a^bE and $\mathfrak{L}_a^bE$ are both free $R-$modules, with

(1.10)
$$\operatorname{rank} \mathfrak{L}_a^bE = \binom{e+a-1}{a+b}\binom{a+b-1}{a-1} \qquad \text{provided } 1 \leq a \text{ and } 0 \leq b \leq e, \text{ and}$$

(1.11)
$$\operatorname{rank} K_a^bE = \binom{e+a-1-b}{a}\binom{e+a}{b} \qquad \text{provided } 0 \leq a \text{ and } 0 \leq b \leq e-1.$$

The isomorphisms of (1.9) are obviously canonical. However, we will, upon occasion, desire a more explicit version of these maps. To that end, consider $\mathbf{s}$ and $\mathbf{t}$ to be homotopies with the properties listed in Proposition 1.6. One quickly sees that

$$S_aE^* \otimes \bigwedge^b E^* = \mathfrak{L}_a^bE^* \oplus \mathbf{s}\mathfrak{L}_{a+1}^{b-1}E^* \quad \text{and} \quad D_aE \otimes \bigwedge^b E^* = K_a^bE \oplus \mathbf{t}K_{a-1}^{b-1}E \tag{1.12}$$

for all integers a and b. If $(a,b) \neq (0,e)$ and $(a,b) \neq (-1,e-1)$, then we see that the isomorphism of (1.9) is actually induced by the perfect pairing

$$\mathfrak{L}_{a+1}^{e-b-1}E^* \otimes K_a^bE \to R \tag{1.13}$$

which sends $x \otimes y$ to $\ll \mathbf{s}(x), y \gg = \pm \ll x, \mathbf{t}(y) \gg$ where $\ll , \gg$ is the perfect pairing of (1.3). The following reformulation of (1.12) will be used later in the paper. If a and b are integers with $(a,b) \neq (-1,1)$, then

$$0 \to \mathfrak{L}_a^bE \xrightarrow{\text{incl}} S_aE \otimes \bigwedge^b E \xrightarrow{\partial_{\text{id}}} \mathfrak{L}_{a+1}^{b-1}E \to 0 \tag{1.14}$$

is a split exact sequence of free $R-$modules.

We conclude our discussion of multilinear algebra by recording a few observations about the interaction between multiple Koszul and Eagon-Northcott maps. These observations are used when we see that the maps in the complexes $\mathfrak{B}^i$ and $\mathfrak{b}^i$ are well defined. If $\rho\colon E \to G$ and $\sigma\colon F \to G$ are maps of free $R-$modules, then it is easy to see that the diagrams

$$\begin{array}{ccc} S_aG \otimes \bigwedge^b E \otimes \bigwedge^c F & \xrightarrow{\partial_\sigma} & S_{a+1}G \otimes \bigwedge^b E \otimes \bigwedge^{c-1} F \\ \partial_\rho \big\downarrow & & \partial_\rho \big\downarrow \\ S_{a+1}G \otimes \bigwedge^{b-1} E \otimes \bigwedge^c F & \xrightarrow{\partial_\sigma} & S_{a+2}G \otimes \bigwedge^{b-1} E \otimes \bigwedge^{c-1} F \end{array} \tag{1.15}$$

and

$$\begin{array}{ccc} D_aG^* \otimes \bigwedge^b E \otimes \bigwedge^c F & \xrightarrow{\delta_\sigma} & D_{a-1}G^* \otimes \bigwedge^b E \otimes \bigwedge^{c-1} F \\ \delta_\rho \big\downarrow & & \delta_\rho \big\downarrow \\ D_{a-1}G^* \otimes \bigwedge^{b-1} E \otimes \bigwedge^c F & \xrightarrow{\delta_\sigma} & D_{a-2}G^* \otimes \bigwedge^{b-1} E \otimes \bigwedge^{c-1} F \end{array} \tag{1.16}$$

both commute. There isn't very much to check because diagram (1.16) is the dual of one of the diagrams of the form of (1.15). (The vertical maps in (1.15) really are $\partial_\rho \otimes 1$. The "$\otimes 1$" doesn't make the notation any more clear and would cause a headache if we insisted on using it in the horizontal maps of (1.15). Therefore, we often omit it. Similarly, when we have homotopies and differentials interacting, we will often write $\mathbf{s}$ and $\mathbf{t}$ in place of $\mathbf{s} \otimes 1$ and $\mathbf{t} \otimes 1$. See Remark 5.11 (b).) Furthermore, if $\rho\colon F \to R$ and $\sigma\colon F \to G$ are maps of free $R-$modules, then the diagrams

$$\begin{array}{ccc} S_aG \otimes \bigwedge^c F & \xrightarrow{\partial_\sigma} & S_{a+1}G \otimes \bigwedge^{c-1} F \\ \partial_\rho \big\downarrow & & \partial_\rho \big\downarrow \\ S_aG \otimes \bigwedge^{c-1} F & \xrightarrow{\partial_\sigma} & S_{a+1}G \otimes \bigwedge^{c-2} F \end{array} \tag{1.17}$$

and

$$\begin{array}{ccc} D_aG^* \otimes \bigwedge^c F & \xrightarrow{\delta_\sigma} & D_{a-1}G^* \otimes \bigwedge^{c-1} F \\ \partial_\rho \big\downarrow & & \partial_\rho \big\downarrow \\ D_aG^* \otimes \bigwedge^{c-1} F & \xrightarrow{\delta_\sigma} & D_{a-1}G^* \otimes \bigwedge^{c-2} F \end{array} \tag{1.18}$$

both commute (up to sign). The second diagram is the dual of one of the diagrams of the form of (1.17) because the Koszul algebra $(S_\bullet R \otimes \bigwedge^\bullet F, \partial_\rho)$ is exactly the same as the Eagon-Northcott algebra $(D_\bullet R^* \otimes \bigwedge^\bullet F, \delta_\rho)$.

If $\Upsilon\colon F \to G$ is a map of free $R-$modules, then $I_t(\Upsilon)$ is the $R-$ideal generated by the $t \times t$ minors of some matrix representation Y of Υ. (The ideal $I_t(\Upsilon)$ does not depend on the choice of Y.) We take $I_0(\Upsilon) = R$ and $I_t(\Upsilon) = (0)$ for $t > \min\{\operatorname{rank} F, \operatorname{rank} G\}$.

The *grade* of a proper ideal I in a ring R is the length of the longest regular sequence on R in I. An $R-$module M is called *perfect* if its projective dimension is equal to the grade of its annihilator. An $R-$ideal I is called *perfect* if R/I is a perfect $R-$module. The following result, which may be proved directly or deduced from [26, Corollary 1.26] together with [9, Proposition 16.33], is an example of how the notion of perfection imitates the notion of Cohen-Macaulay.

Observation 1.19. *If I is a perfect ideal in R and M is a non-zero finitely generated $(R/I)-$module with $\operatorname{pd}_R M \le \operatorname{grade} I$, then M is a perfect $R-$module and every regular sequence on R/I is a regular sequence on M.* $\square$

Section 2. The complex $\mathfrak{B}^i$.

Throughout this section the following data is in effect.

Data 2.1. *The free $R-$modules F and G have rank $f \ge 0$ and $g \ge 1$, respectively,*

$$F \xrightarrow{\Upsilon} G \qquad \text{and} \qquad G \xrightarrow{\Xi} R$$

are $R-$module homomorphisms, K_a^b means $K_a^b G^*$, and $\mathfrak{L}_a^b$ means $\mathfrak{L}_a^b G$.

For each integer i, we consider a bicomplex $\mathbf{B}^i(\Xi,\Upsilon)$. The complex $\mathfrak{B}^i(\Xi,\Upsilon)$, which was advertised below (0.13), is the total complex of $\mathbf{B}^i(\Xi,\Upsilon)$. We write $\mathbf{B}^i$ and $\mathfrak{B}^i$ in place of $\mathbf{B}^i(\Xi,\Upsilon)$ and $\mathfrak{B}^i(\Xi,\Upsilon)$ when it is clear what data is being used. We begin by exhibiting the modules which comprise $\mathbf{B}^i$.

(2.2) The portrait of $\mathbf{B}^i$:

$$\begin{array}{ccccccccc}
K^{g-1}_{f-g-i}\otimes\bigwedge^f F & \to\dots\to & K_0^{g-1}\otimes\bigwedge^{g+i}F & \to & \bigwedge^{i+g-1}F & \to & \mathfrak{L}_1^{g-1}\otimes\bigwedge^{i-1}F & \to\dots\to & \mathfrak{L}_i^{g-1}\otimes\bigwedge^0 F\\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow\\
K^{g-2}_{f-g-i}\otimes\bigwedge^f F & \to\dots\to & K_0^{g-2}\otimes\bigwedge^{g+i}F & \to & \bigwedge^{i+g-2}F & \to & \mathfrak{L}_1^{g-2}\otimes\bigwedge^{i-1}F & \to\dots\to & \mathfrak{L}_i^{g-2}\otimes\bigwedge^0 F\\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow\\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots\\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow\\
K^{1}_{f-g-i}\otimes\bigwedge^f F & \to\dots\to & K_0^{1}\otimes\bigwedge^{g+i}F & \to & \bigwedge^{i+1}F & \to & \mathfrak{L}_1^{1}\otimes\bigwedge^{i-1}F & \to\dots\to & \mathfrak{L}_i^{1}\otimes\bigwedge^0 F\\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow\\
K^{0}_{f-g-i}\otimes\bigwedge^f F & \to\dots\to & K_0^{0}\otimes\bigwedge^{g+i}F & \to & \bigwedge^{i}F & \to & \mathfrak{L}_1^{0}\otimes\bigwedge^{i-1}F & \to\dots\to & \mathfrak{L}_i^{0}\otimes\bigwedge^0 F
\end{array}$$

The module $\bigwedge^i F$ is considered to be $\mathbf{B}^i_{0\,i}$. In other words,

(2.3)
$$\mathbf{B}^i_{a\,b} = \begin{cases} \mathfrak{L}^a_{i-b}\otimes\bigwedge^b F, & \text{if } 0\le a\le g-1 \text{ and } 0\le b\le i-1,\\ \bigwedge^{a+i} F, & \text{if } 0\le a\le g-1 \text{ and } b=i,\\ K^a_{b-i-1}\otimes\bigwedge^{b+g-1}F, & \text{if } 0\le a\le g-1 \text{ and } i+1\le b\le f-g+1.\end{cases}$$

We take $\mathbf{B}^i_{a\,b}$ to be 0 if

$$a\le -1 \quad\text{or}\quad g\le a \quad\text{or}\quad b<\min\{0,i\} \quad\text{or}\quad \max\{i,f-g+1\}<b. \tag{2.4}$$

The horizontal maps in $\big(\mathbf{B}^i,d\big)$ are given by :

$$\begin{array}{ccc}
K_a^b\otimes\bigwedge^c F & \overset{d}{\dashrightarrow} & K^b_{a-1}\otimes\bigwedge^{c-1}F\\
\cap\shortparallel & & \cap\shortparallel\\
D_aG^*\otimes\bigwedge^b G\otimes\bigwedge^c F & \xrightarrow{\delta_\Upsilon} & D_{a-1}G^*\otimes\bigwedge^b G\otimes\bigwedge^{c-1}F,
\end{array} \tag{2.5}$$

$$K_0^b \otimes \bigwedge^{g+i} F = \bigwedge^{b} G \otimes \bigwedge^{g+i} F \xrightarrow{1\otimes\Delta} \bigwedge^{b} G \otimes \bigwedge^{g-b} F \otimes \bigwedge^{i+b} F \xrightarrow{1\otimes\bigwedge^{g-b}\Upsilon\otimes 1} \bigwedge^{b} G \otimes \bigwedge^{g-b} G \otimes \bigwedge^{i+b} F \xrightarrow{\mu\otimes 1} \bigwedge^{i+b} F, \tag{2.6}$$

$$\bigwedge^{i+b} F \xrightarrow{\Delta} \bigwedge^{b+1} F \otimes \bigwedge^{i-1} F \xrightarrow{\bigwedge^{b+1}\Upsilon} \bigwedge^{b+1} G \otimes \bigwedge^{i-1} F \xrightarrow{\partial_{\mathrm{id}}\otimes 1} \mathfrak{L}_1^b \otimes \bigwedge^{i-1} F, \text{ and} \tag{2.7}$$

$$\begin{tikzcd}
\mathfrak{L}_a^b \otimes \bigwedge^c F \arrow[r, dashed, "d"] \arrow[d, phantom, "\cap|"] & \mathfrak{L}_{a+1}^b \otimes \bigwedge^{c-1} F \arrow[d, phantom, "\cap|"] \\
S_aG \otimes \bigwedge^b G \otimes \bigwedge^c F \arrow[r, "\partial_\Upsilon"] & S_{a+1}G \otimes \bigwedge^b G \otimes \bigwedge^{c-1} F.
\end{tikzcd} \tag{2.8}$$

The induced maps of (2.5) and (2.8) exist because of the commutative diagrams (1.16) and (1.15). The vertical maps in $\big(\mathbf{B}^i, d\big)$ are given by:

$$\begin{tikzcd}
K_a^b \otimes \bigwedge^c F \arrow[d, dashed, "d"] & \subseteq & D_aG^* \otimes \bigwedge^b G \otimes \bigwedge^c F \arrow[d, "1\otimes\partial_\Xi\otimes 1"'] \\
K_a^{b-1} \otimes \bigwedge^c F & \subseteq & D_aG^* \otimes \bigwedge^{b-1} G \otimes \bigwedge^c F,
\end{tikzcd} \tag{2.9}$$

$$\begin{tikzcd}
\bigwedge^c F \arrow[d, "\partial_{\Xi\Upsilon}"'] \\
\bigwedge^{c-1} F,
\end{tikzcd} \tag{2.10}$$

and

$$\begin{tikzcd}
\mathfrak{L}_a^b \otimes \bigwedge^c F \arrow[d, dashed, "d"] & \subseteq & S_aG \otimes \bigwedge^b G \otimes \bigwedge^c F \arrow[d, "1\otimes\partial_\Xi\otimes 1"'] \\
\mathfrak{L}_a^{b-1} \otimes \bigwedge^c F & \subseteq & S_aG \otimes \bigwedge^{b-1} G \otimes \bigwedge^c F.
\end{tikzcd} \tag{2.11}$$

The induced maps of (2.9) and (2.11) exist because of the commutative diagrams (1.18) and (1.17). In Proposition 2.18 we prove that $\mathfrak{B}^i$ is a complex. The next result is important in its own right (compare with (0.10)) and it cuts the proof of Proposition 2.18 in half.

Proposition 2.12. *If the data of (2.1) is adopted, then* $\mathfrak{B}^{f-g+1-i} \cong (\mathfrak{B}^i)^*[-f]$.

Proof. A routine calculation using (2.3), (2.4), and (1.9) shows that

$$\mathbf{B}_{a\,b}^{f-g+1-i} \cong \big(\mathbf{B}^i_{g-1-a\ f-g+1-b}\big)^*$$

is a module isomorphism for all integers a, b, and i. It follows that

$$\mathfrak{B}_j^{f-g+1-i} \cong \left(\mathfrak{B}_{f-j}^i\right)^* \tag{2.13}$$

is a module isomorphism for all integers i and j. The duality among the vertical maps follows from the self duality of the Koszul complex. In order to compare the horizontal maps, we fix a pair of families of homotopies:

$$\mathbf{s}\colon S_aG \otimes \bigwedge^b G \to S_{a-1}G \otimes \bigwedge^{b+1} G \quad \text{and} \quad \mathbf{t}\colon D_aG^* \otimes \bigwedge^b G \to D_{a+1}G^* \otimes \bigwedge^{b+1} G$$

which satisfy the properties of Proposition 1.6.

If a and b are integers with $0 \le b \le g-1$ and $1 \le a$, then the diagram

$$\begin{array}{ccc} \mathfrak{L}_a^b \otimes \bigwedge^c F & \xrightarrow{\partial_\Upsilon} & \mathfrak{L}_{a+1}^b \otimes \bigwedge^{c-1} F \\ {\scriptstyle\cong}\big\downarrow & & {\scriptstyle\cong}\big\downarrow \\ \left(K_{a-1}^{g-b-1} \otimes \bigwedge^{f-c} F\right)^* & \xrightarrow{(\delta_\Upsilon)^*} & \left(K_a^{g-b-1} \otimes \bigwedge^{f-c+1} F\right)^*, \end{array} \tag{2.14}$$

with vertical maps given in (1.13) and (1.2), commutes because

$$\ll \partial_\Upsilon x, \mathbf{t}(y) \gg = \pm \ll x, \mathbf{t}\,\delta_\Upsilon y \gg \tag{2.15}$$

for all $x \in \mathfrak{L}_a^b \otimes \bigwedge^c F$ and $y \in K_a^{g-b-1} \otimes \bigwedge^{f-c+1} F$. Indeed, (1.4) and Proposition 1.6 (c) yield that the left side of (2.15) is equal to

$$\begin{aligned} &\pm \ll x, \delta_\Upsilon \mathbf{t}(y) \gg = \pm \ll x, (\delta_{\mathrm{id}}\,\mathbf{t} + \mathbf{t}\,\delta_{\mathrm{id}})\,\delta_\Upsilon \mathbf{t}(y) \gg \\ &= \pm \ll \partial_{\mathrm{id}}\, x, \mathbf{t}\,\delta_\Upsilon \mathbf{t}(y) \gg \pm \ll x, \mathbf{t}\,\delta_{\mathrm{id}}\,\delta_\Upsilon \mathbf{t}(y) \gg . \end{aligned}$$

The domain of x guarantees that $\partial_{\mathrm{id}}\, x = 0$. We know, from (1.16), that the maps δ_{id} and δ_Υ commute; furthermore, $\delta_{\mathrm{id}}\,\mathbf{t}(y) = y - \mathbf{t}\,\delta_{\mathrm{id}}(y) = y$. (The last equality is due to the domain of y.)

If $1 \le i$ and $0 \le b \le g-1$, then we prove that

$$\begin{array}{ccc} \bigwedge^{i+b} F & \xrightarrow{d} & \mathfrak{L}_1^b \otimes \bigwedge^{i-1} F \\ {\scriptstyle\cong}\big\downarrow & & {\scriptstyle\cong}\big\downarrow \\ \left(\bigwedge^{f-i-b} F\right)^* & \xrightarrow{d^*} & \left(K_0^{g-b-1} \otimes \bigwedge^{f-i+1} F\right)^* \end{array} \tag{2.16}$$

commutes by showing that

$$\ll \mathbf{s}dx, y \gg = \pm[x \wedge dy] \tag{2.17}$$

for all $x \in \bigwedge^{i+b} F$ and $y \in \bigwedge^{g-b-1} G \otimes \bigwedge^{f-i+1} F$. Use (2.7) to see that

$$\mathbf{s}dx = \mathbf{s}\,\partial_{\mathrm{id}} \left(\bigwedge^{b+1} \Upsilon\right) \mathit{\Delta} x = \left(\bigwedge^{b+1} \Upsilon\right) \mathit{\Delta} x.$$

(The last equality used the hypothesis that $1 \le b+1$.) One should probably use bases to establish (2.17). When one does this, it is soon clear that only one situation is interesting; namely: $x = w\wedge x'$ and $y = z\otimes w\wedge y'$ for some $w \in \bigwedge^{b+1} F$, $x' \in \bigwedge^{i-1} F$, $y' \in \bigwedge^{f-b-i} F$, and $z \in \bigwedge^{g-b-1} G$. In this case,

$$\ll \mathbf{s}dx, y \gg = \left[\left(\bigwedge^{b+1} \Upsilon\right) w \wedge z\right] [x' \wedge w \wedge y'] \quad \text{and}$$

$$[x \wedge dy] = \left[z \wedge \left(\bigwedge^{b+1} \Upsilon\right) w\right] [w \wedge x' \wedge y']. \quad \square$$

Proposition 2.18. *If the data of (2.1) is adopted, then $\mathfrak{B}^i$ is a complex.*

Proof. It suffices to prove that $\mathbf{B}^i$ is a bicomplex. Most of the maps of $\mathbf{B}^i$ are induced by Koszul algebra maps or Eagon-Northcott algebra maps. Since the graded strands of these algebras are complexes, we know that all of the columns of $\mathbf{B}^i$ are complexes and all of the rows of $\mathbf{B}^i$ are complexes except, possibly, near the i^{th} position. The sequence

$$\bigwedge^{i+b} F \to \mathfrak{L}_1^b \otimes \bigwedge^{i-1} F \to \mathfrak{L}_2^b \otimes \bigwedge^{i-2} F \tag{2.19}$$

is a complex because Δ is co-associative and the composition

$$\bigwedge^{2} F \xrightarrow{\Delta} F \otimes F \xrightarrow{\mu} S_2F$$

is zero. The sequence

$$K_1^b \otimes \bigwedge^{g+i+1} F \to K_0^b \otimes \bigwedge^{g+i} F \to \bigwedge^{i+b} F$$

is also a complex because we know, from Proposition 2.12, that it is the dual of the complex which is obtained from (2.19) when i is replaced by $f-g+1-i$ and b is replaced by $g-b-1$. The composition

$$K_0^b \otimes \bigwedge^{g+i} F \xrightarrow{d} \bigwedge^{i+b} F \xrightarrow{d} \mathfrak{L}_1^b \otimes \bigwedge^{i-1} F$$

with $1 \le i$ factors through the composition

$$\bigwedge^{b} G \otimes \bigwedge^{g+i} F \xrightarrow{1\otimes\Delta} \bigwedge^{b} G \otimes \bigwedge^{g+1} F \otimes \bigwedge^{i-1} F \xrightarrow{1\otimes\bigwedge^{g+1}\Upsilon\otimes 1} \bigwedge^{b} G \otimes \bigwedge^{g+1} G \otimes \bigwedge^{i-1} F = 0;$$

and therefore is zero.

The squares in $\mathbf{B}^i$ of the form

$$\begin{array}{ccc} \mathfrak{L}_a^b \otimes \bigwedge^c F & \longrightarrow & \mathfrak{L}_{a+1}^b \otimes \bigwedge^{c-1} F \\ \downarrow & & \downarrow \\ \mathfrak{L}_a^{b-1} \otimes \bigwedge^c F & \longrightarrow & \mathfrak{L}_{a+1}^{b-1} \otimes \bigwedge^{c-1} F \end{array}$$

commute because, as we observed in (1.15), the Koszul differentials ∂_Υ and ∂_Ξ commute. The proof of Proposition 3.2 in [8] explains why the squares

$$\begin{array}{ccc} K_0^b \otimes \bigwedge^{g+i} F & \longrightarrow & \bigwedge^{i+b} F \\ \big\downarrow & & \big\downarrow \\ K_0^{b-1} \otimes \bigwedge^{g+i} F & \longrightarrow & \bigwedge^{i+b-1} F \end{array}$$

commute (up to sign). The duality guaranteed by Proposition 2.12 yields that the other two types of squares from $\mathbf{B}^i$ also commute (up to sign). $\square$

Section 3. Elementary facts about the complexes $\mathfrak{B}^i$.

In this section we collect a few facts which can be read directly from the definition of the complex $\mathfrak{B}^i$. We begin by recording some qualitative data about the length, Betti numbers, and twists of each $\mathfrak{B}^i$. For example, if $g-1 \le f$ and $f-g+2 \le i \le f+1$, then a quick look at (2.2) shows that $\mathfrak{B}^i_j \ne 0$ if and only if $0 \le j \le g+i-2$. One can use (2.13) to learn similar information for $-g \le i \le -1$. The best way to record the length of $\mathfrak{B}^i$ is by using a graph.

Observation 3.1. *If the notation of (2.1) is adopted, then the module $\mathfrak{B}^i_j \ne 0$ if and only if there is a dot at the point (i,j) in Figure 3.2 or Figure 3.3.*

Pairs (i,j) with $\mathfrak{B}^i_j \ne 0$ when $g-1 \le f$:

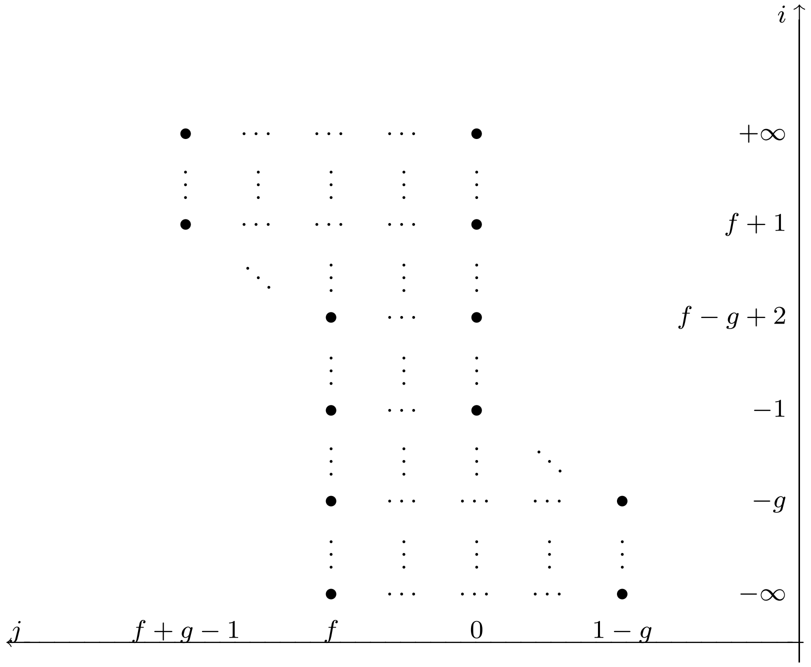


Figure 3.2

The betti numbers of $\mathfrak{B}^i$ may be easily read from (1.10) and (1.11). We next describe the twists in the complex $\mathfrak{B}^i$. In the notation of (2.1) consider R to be a graded ring and view Ξ and Υ as matrices of one forms from R. Each differential

Pairs (i,j) with $\mathfrak{B}^i_j \neq 0$ when $f \le g-1$:

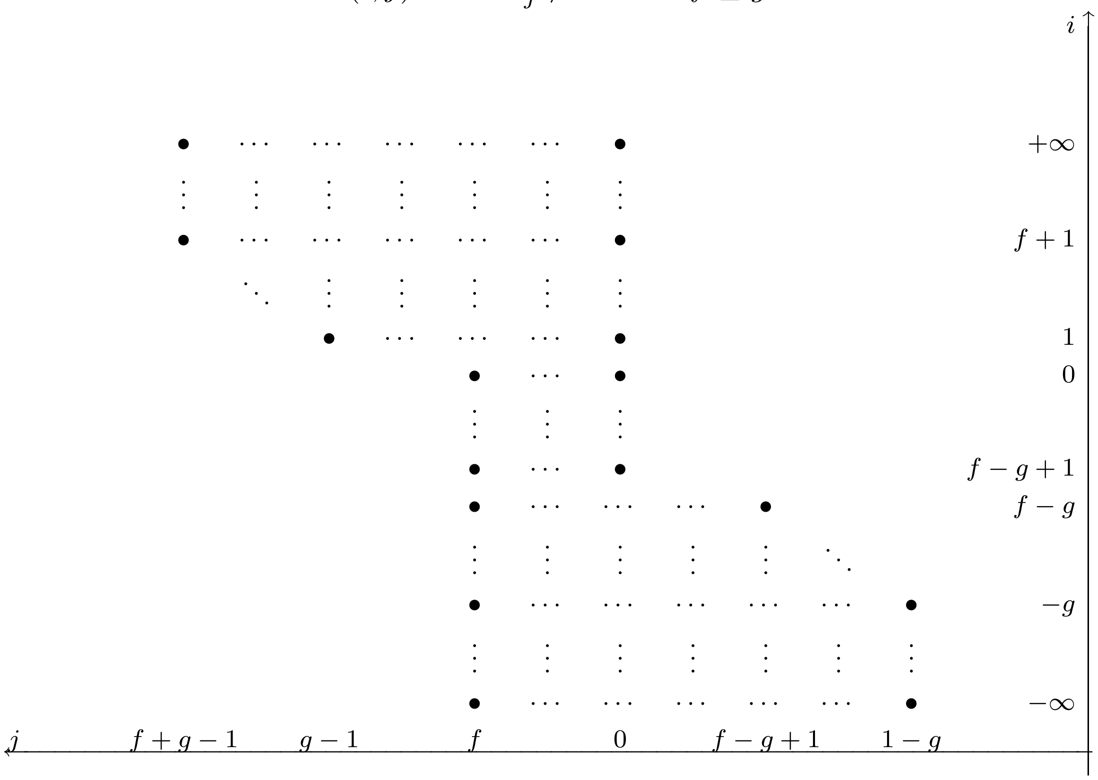


Figure 3.3

The degree of the maps in $\mathfrak{B}^i$:

$$\begin{tikzcd}
K^{g-1\,f}_{f-g-i} \arrow[r, "1"] \arrow[d, "1"'] & \cdot \arrow[r, "1"] & K_0^{g-1\,g+i} \arrow[r, "1"] \arrow[d, "1"'] & \bigwedge^{i+g-1} \arrow[r, "g"] \arrow[d, "2"'] & \mathfrak{L}_1^{g-1\,i-1} \arrow[r, "1"] \arrow[d, "1"'] & \cdot \arrow[r, "1"] & \mathfrak{L}_i^{g-1\,0} \arrow[d, "1"'] \\
K^{g-2\,f}_{f-g-i} \arrow[r, "1"] \arrow[d, "1"'] & \cdot \arrow[r, "1"] & K_0^{g-2\,g+i} \arrow[r, "2"] \arrow[d, "1"'] & \bigwedge^{i+g-2} \arrow[r, "g-1"] \arrow[d, "2"'] & \mathfrak{L}_1^{g-2\,i-1} \arrow[r, "1"] \arrow[d, "1"'] & \cdot \arrow[r, "1"] & \mathfrak{L}_i^{g-2\,0} \arrow[d, "1"'] \\
\vdots \arrow[d, "1"'] & & \vdots \arrow[d, "1"'] & \vdots \arrow[d, "2"'] & \vdots \arrow[d, "1"'] & & \vdots \arrow[d, "1"'] \\
K^{1\,f}_{f-g-i} \arrow[r, "1"] \arrow[d, "1"'] & \cdot \arrow[r, "1"] & K_0^{1\,g+i} \arrow[r, "g-1"] \arrow[d, "1"'] & \bigwedge^{i+1} \arrow[r, "2"] \arrow[d, "2"'] & \mathfrak{L}_1^{1\,i-1} \arrow[r, "1"] \arrow[d, "1"'] & \cdot \arrow[r, "1"] & \mathfrak{L}_i^{1\,0} \arrow[d, "1"'] \\
K^{0\,f}_{f-g-i} \arrow[r, "1"] & \cdot \arrow[r, "1"] & K_0^{0\,g+i} \arrow[r, "g"] & \bigwedge^{i} \arrow[r, "1"] & \mathfrak{L}_1^{0\,i-1} \arrow[r, "1"] & \cdot \arrow[r, "1"] & \mathfrak{L}_i^{0\,0}
\end{tikzcd}$$

Figure 3.4

map in $\mathbf{B}^i$ can be viewed as a matrix of homogeneous forms from R. In this language, maps (2.8), (2.9), and (2.11) are matrices of linear forms; map (2.10) is a matrix of quadratic forms; map (2.6) is a matrix of forms of degree $g-b$; and map (2.7) is a matrix of forms of degree $b+1$. In Figure 3.4 we have recorded the degree of the entries of each map from $\mathbf{B}^i$. For space reasons we have written $\mathfrak{L}_a^{b\,c}$, $K_a^{b\,c}$, and $\bigwedge^c$ in place of $\mathfrak{L}_a^b \otimes \bigwedge^c F$, $K_a^b \otimes \bigwedge^c F$, and $\bigwedge^c F$, respectively. It is not difficult to translate Figure 3.4 into the usual notation for twists. For example, if $0 \le i$, $g+i \le f$, and $2 \le g$, then $\mathfrak{B}_0^i = \mathfrak{L}_i^{0\,0} = R^n$ for $n = \binom{g+i-1}{i}$ and the summand $K_0^1 \otimes \bigwedge^{g+i} F$ of $\mathfrak{B}_{i+2}^i$ is $R^m(-(g+i+1))$ for $m = g\binom{f}{g+i}$.

Next, we record the zeroth homology of those complexes $\mathfrak{B}^i$ which begin at position zero. The following notation is in effect.

Notation 3.5. *Retain the data of (2.1). Let $\mathcal{J}(\Xi,\Upsilon)$ be the ideal $I_1(\Xi\Upsilon)+I_g(\Upsilon)$ of R, $A(\Xi,\Upsilon)$ be the ring $R/\mathcal{J}(\Xi,\Upsilon)$, and $M(\Xi,\Upsilon)$ be the $R-$module presented by*

$$[\partial_\Xi\ \Upsilon]\colon \bigwedge^2 G \oplus F \to G.$$

When there is no ambiguity, we write A, $\mathcal{J}$, and M in place of $A(\Xi,\Upsilon)$, $\mathcal{J}(\Xi,\Upsilon)$, and $M(\Xi,\Upsilon)$. If $g \le f$, then $\mathcal{J}$ is one of the ideals "J" from (0.1); otherwise, $\mathcal{J}$ is simply equal to $I_1(\Xi\Upsilon)$. It is easy to see that M is actually an $A-$module. There are several interpretations of the $A-$module M.

Observation 3.6. *Adopt the notation of (3.5).*

(a) *There is a surjection from M onto the $A-$ideal*

$$\frac{I_1(\Xi)+\mathcal{J}}{\mathcal{J}}.$$

(b) *Let $\varphi_1,\dots,\varphi_f$ and $\gamma_1,\dots,\gamma_g$ be bases for F and G, respectively; and let X and Y be the matrices of Ξ and Υ with respect to these bases. If $g-1 \le f$, then there is a surjection from M onto the $A-$ideal*

$$\frac{I_{g-1}(\textit{columns } 1 \textit{ to } g-1 \textit{ of } Y)+\mathcal{J}}{\mathcal{J}}.$$

(c) *If* $\operatorname{grade} I_1(\Xi)=g$, *then* $M \cong I_1(\Xi)/I_1(\Xi\Upsilon)$.

Note. If the data of (2.1) is sufficiently general (see Theorem 9.2 (a) and (10.16)), then the surjections of (a) and (b) are isomorphisms.

Observation 3.7. *In the notation of (3.5) the following statements hold.*

(a)

$$H_0(\mathfrak{B}^i) = \begin{cases} A, & \textit{if } i=0, \textit{ or if } f-g+1 \le i \le -1, \textit{ and} \\ S_i(M), & \textit{if } i>0. \end{cases}$$

(b) *Let γ be any element of G, $\Upsilon_\gamma\colon F\oplus R \to G$ be the extension of Υ which sends $\begin{bmatrix}0\\1\end{bmatrix}$ to γ, and $\mathcal{J}_\gamma$ be the ideal $\mathcal{J}(\Xi,\Upsilon_\gamma)/\mathcal{J}$ of A. Then there is an $R-$module surjection $H_0(\mathfrak{B}^{-1}) \twoheadrightarrow \mathcal{J}_\gamma$.*

Note. If $\gamma=\gamma_g$, in the notation of Observation 3.6 (b), then the ideal $\mathcal{J}_\gamma$ of Observation 3.7 (b) is

$$\frac{(x_g)+I_{g-1}(\text{rows } 1 \text{ to } g-1 \text{ of } Y)+\mathcal{J}}{\mathcal{J}}. \tag{3.8}$$

If the data of (2.1) is sufficiently general (see Theorems 9.2 (a) and 9.3 (d), Remark 9.5 (a), and (10.15)), then the surjection of (b) is an isomorphism.

Proof. The proof of (a) is straightforward. We prove (b). The R–module $H_0(\mathfrak{B}^{-1})$ is presented by

$$\begin{bmatrix} \delta\Upsilon & \Xi\otimes 1 & 0 \\ 0 & -\mu(1\otimes\bigwedge^{g-1}\Upsilon) & \Xi\Upsilon \end{bmatrix} : \left(G^*\otimes\bigwedge^{g}F\right)\oplus\left(G\otimes\bigwedge^{g-1}F\right)\oplus F\to\bigwedge^{g-1}F\oplus R,$$

and the map $\bigwedge^{g-1}F\oplus R\to\mathcal{J}_\gamma$, which is given by

$$\begin{bmatrix} x \\ r \end{bmatrix}\longmapsto\left[\gamma\wedge(\bigwedge^{g-1}\Upsilon)(x)\right]+r\Xi(\gamma),$$

induces the desired surjection. $\square$

It is important, for the purposes of induction, that we understand the complex $\mathfrak{B}^i$ whenever either of the parameters f or g is small.

Example 3.9. If $g=1$, then $\mathfrak{B}^i$ is the bottom row of $\mathbf{B}^i$ for all integers i. Furthermore, in this case, $\mathfrak{B}^i$ is the usual Koszul complex associated to the map $\Upsilon\colon F\to R$.

If $f=0$, then the complex $\mathfrak{B}^i$ is also well understood. Indeed, if $1-g\le i\le 0$, then $\mathfrak{B}^i$ consists of the module R concentrated in position zero. Furthermore, if $i\ge 1$, then $\mathfrak{B}^i$ consists of the right most column from $\mathbf{B}^i$ and it is essentially the same as the complex $\mathbf{L}^1_i(\Xi)$ of [11, Corollary 3.2]. (Srinivasan [30, Theorem 2.1] referred to $\mathbf{L}^1_i(\Xi)$ as $L^i(\Xi)$, and she proved that $\mathbf{L}^1_i(\Xi)$ admits the structure of a DG–algebra.) At any rate, $\mathbf{L}^1_i(\Xi)$ is

$$\underbrace{0\to\mathfrak{L}_i^{g-1}\to\cdots\to\mathfrak{L}_i^0}_{\mathfrak{B}^i}=S_iG\xrightarrow{S_i(\Xi)}R,$$

and we draw the following conclusion.

Observation 3.10. *Adopt the data of (2.1) with $f=0$. If* $\operatorname{grade} I_1(\Xi)=g$, *then $\mathfrak{B}^i$ is a resolution of $S_i(I_1(\Xi))\cong(I_1(\Xi))^i$ for all $i\ge 1$.*

One facet of the proof that the $\mathfrak{B}^i$ are acyclic (see section 4) entails inverting a minor of Ξ or Υ. We determine the effect of such localizations in the following two examples.

Example 3.11. If the ring R in the notation of (3.5) is local and $I_1(\Xi)=R$, then $H_0(\mathfrak{B}^i)=R/\mathcal{J}$ for all $i\ge 0$. Indeed, one can choose bases for F and G so that the matrix of Ξ is $[1\quad 0\quad\dots\quad 0]$.

Example 3.12. In the notation of (3.5), suppose that T is an indeterminate over R. Let

$$\mathcal{X}=[\Xi\quad T],\quad\mathcal{Y}=\begin{bmatrix}\Upsilon & 0\\ 0 & 1\end{bmatrix},\quad\text{and}\quad\mathcal{M}=M(\mathcal{X},\mathcal{Y}).$$

It is clear that the $R[T]-$ideal $\mathcal{J}(\mathcal{X},\mathcal{Y})$ is equal to $\mathcal{J}+(T)$. Furthermore, if the $R-$module M is viewed as an $R[T]-$module by way of the ring homomorphism

$$R[T]\to R[T]/(T)=R,$$

then $\mathcal{M}$ and M are isomorphic as $R[T]-$modules. In particular, if $\operatorname{pd}_R S_i(M)<\infty$ for some $i\ge 0$, then $\operatorname{pd}_{R[T]} S_i(\mathcal{M})=\operatorname{pd}_R S_i(M)+1$.

We conclude this section by observing that the complex $\mathfrak{B}^i$ is the mapping cone of two complexes constructed from smaller data; thereby showing that the family $\{\mathfrak{B}^i\}$ satisfies a property analogous to (0.12). In the next result we write $\widetilde{A}$, $\widetilde{\mathfrak{B}}$, $\widetilde{\mathbf{B}}$, $\widetilde{\mathcal{J}}$, and $\widetilde{M}$ for $A(\Xi,\widetilde{\Upsilon})$, $\mathfrak{B}(\Xi,\widetilde{\Upsilon})$, $\mathbf{B}(\Xi,\widetilde{\Upsilon})$, $\mathcal{J}(\Xi,\widetilde{\Upsilon})$, and $M(\Xi,\widetilde{\Upsilon})$.

Proposition 3.13. *In the notation of (2.1), let $F=\widetilde{F}\oplus R\varphi$ for some free $R-$module $\widetilde{F}$ of rank $f-1$, and let $\widetilde{\Upsilon}$ be the restriction of Υ to $\widetilde{F}$. Then, for every integer i, there is a short exact sequence of complexes*

$$0\to\widetilde{\mathfrak{B}}^i\to\mathfrak{B}^i\to\widetilde{\mathfrak{B}}^{i-1}[-1]\to 0; \tag{3.14}$$

in particular, there is a long exact sequence of homology

(3.15)

$$\cdots\to \mathrm{H}_{j+1}(\mathfrak{B}^i)\to H_j(\widetilde{\mathfrak{B}}^{i-1})\to H_j(\widetilde{\mathfrak{B}}^i)\to H_j(\mathfrak{B}^i)\to H_{j-1}(\widetilde{\mathfrak{B}}^{i-1})\to\dots\quad .$$

Furthermore, the sequences

(3.16)

$$H_1(\widetilde{\mathfrak{B}}^i)\to H_1(\mathfrak{B}^i)\to S_{i-1}(\widetilde{M})\xrightarrow{\Upsilon(\varphi)} S_i(\widetilde{M})\to S_i(M)\to 0,\quad \textit{for } i\ge 1, \textit{ and}$$

$$H_1(\widetilde{\mathfrak{B}}^0)\to H_1(\mathfrak{B}^0)\to H_0(\widetilde{\mathfrak{B}}^{-1})\to \mathcal{J}/\widetilde{\mathcal{J}}\to 0 \tag{3.17}$$

are exact, where $S_0(\widetilde{M})$ is taken to mean $\widetilde{A}$.

Proof. Recall the definition of $\mathbf{B}^i_{a\,b}$ given in (2.3). For each triple of integers (a,b,i), the decomposition $\bigwedge^c F=\bigwedge^c\widetilde{F}\oplus\left(\bigwedge^{c-1}\widetilde{F}\otimes R\varphi\right)$ yields a short exact sequence of modules

$$0\to\widetilde{\mathbf{B}}^i_{a\,b}\to\mathbf{B}^i_{a\,b}\to\widetilde{\mathbf{B}}^{i-1}_{a\,b-1}\to 0.$$

It is not difficult to check that these exact sequences induce the short exact sequence (3.14); and therefore, the long exact sequence (3.15). If $0\le i$, then the module $H_0(\mathfrak{B}^i)$ has been identified in Observation 3.7. The exactness of (3.17) follows immediately. The exactness of (3.16) is established as soon as one checks that the connecting homomorphism

$$S_{i-1}(\widetilde{M})\cong H_0(\widetilde{\mathfrak{B}}^{i-1})\to H_0(\widetilde{\mathfrak{B}}^i)\cong S_i(\widetilde{M})$$

is multiplication by the image of the element $\Upsilon(\varphi)$ in the symmetric algebra $S^{\widetilde{A}}_\bullet(\widetilde{M})$. $\square$

Remark. In our proof of Proposition 3.13 the exactness of (3.17) was established by formal considerations. However, we could have appealed to Observation 3.7 (b) in order to produce a surjection $H_0(\widetilde{\mathfrak{B}}^{-1})\twoheadrightarrow\mathcal{J}/\widetilde{\mathcal{J}}$ because, if one starts with the map $\widetilde{\Upsilon}\colon\widetilde{F}\to R$ and one chooses γ to be the element $\Upsilon(\varphi)$ of G, then the ideal $\mathcal{J}_\gamma$ of Observation 3.7 (b) is $\mathcal{J}/\widetilde{\mathcal{J}}$.

Section 4. Acyclicity of $\mathfrak{B}^i$ in the generic case.

Notation 4.1. *Let $f \ge 0$ and $g \ge 1$ be integers, R_0 be a commutative noetherian ring, $X_{1\times g} = (x_i)$ and $Y_{g\times f} = (y_{ij})$ be matrices of indeterminates, and R be the polynomial ring $R_0[X, Y]$. View $Y: F \to G$ and $X: G \to R$ as maps of free $R-$modules. Let $\mathfrak{B}$ represent $\mathfrak{B}(X, Y)$, $\mathcal{J}$ be the $R-$ideal $I_1(XY) + I_g(Y)$, and A be the quotient $R/\mathcal{J}$.*

Theorem 4.2. *Adopt the notation of (4.1). If $\min\{-1, f-g+1\} \le i$, then $\mathfrak{B}^i$ is acyclic and $H_0(\mathfrak{B}^i)$ is isomorphic to an ideal of A.*

Proof. If $f-g+1 \le i \le 0$, then $\mathfrak{B}^i$ is the Koszul complex associated to $XY: F \to R$. Since the entries of the matrix XY form a regular sequence whenever $f \le g$, we conclude that $\mathfrak{B}^i$ is acyclic and $H_0(\mathfrak{B}^i) \cong A$. Henceforth, we assume that either $0 \le i$, or else, that $i = -1$ and $g - 1 \le f$. The proof proceeds by induction on f. Observation 3.10 takes care of the case $f = 0$; henceforth, we assume that $1 \le f$. Let $\widetilde{Y}$ represent the submatrix of Y which consists of columns 1 to $f-1$,

$$\widetilde{\mathfrak{B}}^i = \mathfrak{B}^i(X, \widetilde{Y}), \qquad \widetilde{\mathcal{J}} = I_1(X\widetilde{Y}) + I_g(\widetilde{Y}), \qquad \widetilde{A} = R/\widetilde{\mathcal{J}},$$

$\widetilde{I}$ be the $\widetilde{A}-$ideal generated by $I_1(X)$, and z be the element $\sum_{i=1}^{g} x_i y_{if}$ of $\widetilde{I}$. The induction hypothesis guarantees that $\widetilde{\mathfrak{B}}^i$ is acyclic for all $i \ge -1$. Therefore, the long exact sequence (3.15) yields $H_j(\mathfrak{B}^i) = 0$ for all i and j with $j \ge 2$ and $i \ge 0$. Further observations are necessary before we consider $H_1(\mathfrak{B}^i)$. The following statements hold.

(4.3) The $R-$ideal $\widetilde{\mathcal{J}}$ is perfect of grade $f-1$.

(4.4) The $\widetilde{A}-$ideal $\widetilde{I}$ has positive grade.

(4.5) The element z is regular on $\widetilde{A}$.

Indeed, if $f - 1 < g$, then $\widetilde{\mathcal{J}}$ is generated by a regular sequence; and if $g \le f-1$, then a proof of (4.3) is contained in [8, Proposition 4.2], see (0.2). Assertion (4.4) follows from (4.3) because $f \le \operatorname{grade}\left(I_1(X) + I_g(\widetilde{Y})\right)$. Hochster's notion of general grade reduction [14] ensures (4.5).

We saw in Observations 3.7 (a) and 3.6 (a) that there is an $\widetilde{A}-$module surjection

$$H_0(\widetilde{\mathfrak{B}}^i) = S_i(\widetilde{M}) \twoheadrightarrow \widetilde{I}^i \tag{4.6}$$

for all $i \ge 0$. Since $\widetilde{I}$ has positive grade (by (4.4)), and $H_0(\widetilde{\mathfrak{B}}^i)$ is isomorphic to an ideal of $\widetilde{A}$ (by induction), we conclude that (4.6) is an isomorphism. When the isomorphism of (4.6) is applied to the exact sequence (3.16) the image of "$\Upsilon(\varphi)$" in $\widetilde{I}$ is z; consequently,

$$0 = H_1(\widetilde{\mathfrak{B}}^i) \to H_1(\mathfrak{B}^i) \to \widetilde{I}^{i-1} \xrightarrow{z} \widetilde{I}^i$$

is exact. Use (4.5) in order to conclude that $H_1(\mathfrak{B}^i) = 0$ for all $i \ge 1$.

The ideal $\mathcal{J}$ also contains z; consequently, the same argument as above yields that the surjection $H_0(\widetilde{\mathfrak{B}}^{-1}) \twoheadrightarrow \mathcal{J}/\widetilde{\mathcal{J}}$ of (3.17) is also an isomorphism. It follows that $H_1(\mathfrak{B}^0) = 0$; and therefore, $\mathfrak{B}^i$ is acyclic for all $i \geq 0$. If $g-1 \leq f$, then the complex $\mathfrak{B}^{f-g+2}$ has length f (see Figure 3.2), and it resolves a prefect $R-$module of projective dimension f; thus, $\mathfrak{B}^{-1} \cong \left(\mathfrak{B}^{f-g+2}\right)^* [-f]$ is also acyclic.

It remains to show that $H_0(\mathfrak{B}^i)$ is isomorphic to an ideal of A. Fix $i \geq 1$. It is easy to see that the $A-$module $H_0(\mathfrak{B}^i)$ has rank one. (See [9, Section 16.A] or [26, Observation 1.27] for a discussion of rank.) Indeed, if P is an associated prime of A, then $I_1(X)A \not\subseteq P$ (see (4.4)); hence, Example 3.11 shows that $H_0(\mathfrak{B}^i)_P = A_P$. We next show that $H_0(\mathfrak{B}^i)$ is a torsion-free $A-$module. The annihilator of $H_0(\mathfrak{B}^i)$ contains the ideal $\mathcal{J}$, which is a perfect ideal of grade f. Since $\mathfrak{B}^i$ is acyclic, we know, from Observation 3.1, that

$$f \leq \operatorname{pd}_R H_0(\mathfrak{B}^i) \leq f+g-1.$$

Let $j \geq f+1$ be fixed, and let F_j be the radical of the $R-$ideal generated by

$$\{x \in R \mid \operatorname{pd}_{R_x} H_0(\mathfrak{B}^i)_x < j\}. \tag{4.7}$$

Example 3.11 shows that $I_1(X) \subseteq F_j$. If $j \leq g-1$, then

$$j+1 \leq g \leq \operatorname{grade} F_j.$$

If $g \leq j \leq f+g-1$, then a quick look at Example 3.12 shows that $I_1(X) + I_{f+g-j}(Y) \subseteq F_j$; and therefore,

$$j+2 \leq g+2(j-g+1) \leq g+(j-f+1)(j-g+1) \leq \operatorname{grade} F_j. \tag{4.8}$$

In any event, we see that $j+1 \leq \operatorname{grade} F_j$ for all j with $f+1 \leq j \leq f+g-1$. It follows (see, for example, [26, Proposition 1.25]) that $H_0(\mathfrak{B}^i)$ is a torsion-free $A-$module. We conclude that the surjection

$$\mathrm{H}_0(\mathfrak{B}^i) \twoheadrightarrow I_1(X)^i A \tag{4.9}$$

is an isomorphism. Finally, we consider the case $i=-1$ and $g-1 \leq f$. We have seen that $H_0(\mathfrak{B}^{f-g+2})$ is a perfect $R-$module of projective dimension f and

$$H_0(\mathfrak{B}^{-1}) = \operatorname{Ext}_R^f(H_0(\mathfrak{B}^{f-g+2}), R). \tag{4.10}$$

It follows that $H_0(\mathfrak{B}^{-1})$ is a torsion-free $A-$module. (See, for example, Observation 1.19.) If $P \in \operatorname{Ass}(A)$, then Example 3.11 shows that $H_0(\mathfrak{B}^{f-g+2})_P$ is obtained from R_P by modding out a regular sequence of length f; thus, (4.10) yields that $H_0(\mathfrak{B}^{-1})$ has rank one. Recall, from (3.8), that there is an $A-$module surjection

$$H_0(\mathfrak{B}^{-1}) \twoheadrightarrow \frac{(x_g)+I_{g-1}(\text{rows } 1 \text{ to } g-1 \text{ of } Y)+I_1(XY)}{I_1(XY)+I_g(Y)}. \tag{4.11}$$

Since it is easy to see that the $A-$ideal on the right side of (4.11) has positive grade, it follows that (4.11) is an isomorphism. $\square$

SECTION 5. THE COMPLEX $\mathfrak{b}^i$.

Once again, we begin with the data of (0.13).

Data 5.1. *The free $R-$modules F and G have rank $f \ge 1$ and $g \ge 0$, respectively,*

$$F \xrightarrow{\Upsilon} G \qquad \text{and} \qquad G \xrightarrow{\Xi} R$$

are $R-$module homomorphisms, K_a^b means K_a^bF, and $\mathfrak{L}_a^b$ means $\mathfrak{L}_a^bF^*$. A pair of homotopies

$$\mathbf{s}\colon S_aF^* \otimes \bigwedge^b F^* \to S_{a-1}F^* \otimes \bigwedge^{b+1} F^* \quad \text{and} \quad \mathbf{t}\colon D_aF \otimes \bigwedge^b F^* \to D_{a+1}F \otimes \bigwedge^{b+1} F^*,$$

which satisfy the properties of Proposition 1.6 is fixed. Let ξ represent the element Ξ of G^*, and υ represent the element $\Xi\Upsilon$ of F^*.

(Notice that the abbreviations K_a^b and $\mathfrak{L}_a^b$ have different meanings in (2.1) and (5.1).) For each integer i, we consider a bicomplex $\mathbf{b}^i(\Xi,\Upsilon) = \mathbf{b}^i$. The complex $\mathfrak{b}^i(\Xi,\Upsilon) = \mathfrak{b}^i$ is the total complex of $\mathbf{b}^i$.

(5.2) THE PORTRAIT OF $\mathbf{b}^i$:

$$\begin{array}{ccccccccc}
K^0_{g-f-i-1}\otimes\bigwedge^g G^* \to\dots\to & K^0_0\otimes\bigwedge^{f+i+1}G^* & \to & \bigwedge^{i+1}G^* & \to & \mathfrak{L}^0_1\otimes\bigwedge^i G^* & \to\dots\to & \mathfrak{L}^0_{i+1}\otimes\bigwedge^0 G^* \\
\downarrow & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
K^1_{g-f-i-1}\otimes\bigwedge^g G^* \to\dots\to & K^1_0\otimes\bigwedge^{f+i+1}G^* & \to & \bigwedge^{i+2}G^* & \to & \mathfrak{L}^1_1\otimes\bigwedge^i G^* & \to\dots\to & \mathfrak{L}^1_{i+1}\otimes\bigwedge^0 G^* \\
\downarrow & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\vdots & \vdots & & \vdots & & \vdots & & \vdots \\
\downarrow & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
K^{f-2}_{g-f-i-1}\otimes\bigwedge^g G^* \to\dots\to & K^{f-2}_0\otimes\bigwedge^{f+i+1}G^* & \to & \bigwedge^{f+i-1}G^* & \to & \mathfrak{L}^{f-2}_1\otimes\bigwedge^i G^* & \to\dots\to & \mathfrak{L}^{f-2}_{i+1}\otimes\bigwedge^0 G^* \\
\downarrow & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
K^{f-1}_{g-f-i-1}\otimes\bigwedge^g G^* \to\dots\to & K^{f-1}_0\otimes\bigwedge^{f+i+1}G^* & \to & \bigwedge^{f+i}G^* & \to & \mathfrak{L}^{f-1}_1\otimes\bigwedge^i G^* & \to\dots\to & \mathfrak{L}^{f-1}_{i+1}\otimes\bigwedge^0 G^*
\end{array}$$

The module $\bigwedge^{f+i} G^*$ is considered to be $\mathbf{b}^i_{0\,i+1}$. In other words,

(5.3)
$$\mathbf{b}^i_{a\,b} = \begin{cases} \mathfrak{L}^{f-1-a}_{i+1-b}\otimes\bigwedge^b G^*, & \text{if } 0\le a\le f-1 \text{ and } 0\le b\le i,\\ \bigwedge^{f+i-a}G^*, & \text{if } 0\le a\le f-1 \text{ and } b=i+1,\\ K^{f-1-a}_{b-i-2}\otimes\bigwedge^{b+f-1}G^*, & \text{if } 0\le a\le f-1 \text{ and } i+2\le b\le g-f+1.\end{cases}$$

We take $\mathbf{b}^i_{a\,b}$ to be 0 if

(5.4)
$$a\le -1 \quad \text{or} \quad f\le a \quad \text{or} \quad b<\min\{0,i+1\} \quad \text{or} \quad \max\{i+1,g-f+1\}<b.$$

The horizontal maps in $\left(\mathbf{b}^i, d\right)$ are given by :

(5.5)
$$\begin{array}{ccc}
K_a^b\otimes\bigwedge^c G^* & \dashrightarrow & K_{a-1}^b\otimes\bigwedge^{c-1}G^* \\
\cap\text{||} & & \cap\text{||} \\
D_aF\otimes\bigwedge^b F^*\otimes\bigwedge^c G^* & \xrightarrow{\delta_{\Upsilon^*}} & D_{a-1}F\otimes\bigwedge^b F^*\otimes\bigwedge^{c-1}G^*,
\end{array}$$

$$K_0^b \otimes \bigwedge^{f+c} G^* = \bigwedge^b F^* \otimes \bigwedge^{f+c} G^* \xrightarrow{1\otimes\Delta} \bigwedge^b F^* \otimes \bigwedge^{f-b} G^* \otimes \bigwedge^{b+c} G^*$$
$$\xrightarrow{1\otimes\bigwedge^{f-b}\Upsilon^*\otimes 1} \bigwedge^b F^* \otimes \bigwedge^{f-b} F^* \otimes \bigwedge^{b+c} G^* \xrightarrow{\mu\otimes 1} \bigwedge^{b+c} G^*, \tag{5.6}$$

$$\bigwedge^{b+c} G^* \xrightarrow{\Delta} \bigwedge^{b+1} G^* \otimes \bigwedge^{c-1} G^* \xrightarrow{\bigwedge^{b+1}\Upsilon^*} \bigwedge^{b+1} F^* \otimes \bigwedge^{c-1} G^*$$
$$\xrightarrow{\partial_{\mathrm{id}}\otimes 1} \mathfrak{L}_1^b \otimes \bigwedge^{c-1} G^*, \quad \text{and} \tag{5.7}$$

$$\begin{array}{ccc} \mathfrak{L}_a^b \otimes \bigwedge^c G^* & \dashrightarrow & \mathfrak{L}_{a+1}^b \otimes \bigwedge^{c-1} G^* \\ \cap\shortparallel & & \cap\shortparallel \\ S_aF^* \otimes \bigwedge^b F^* \otimes \bigwedge^c G^* & \xrightarrow{\partial_{\Upsilon^*}} & S_{a+1}F^* \otimes \bigwedge^b F^* \otimes \bigwedge^{c-1} G^*. \end{array} \tag{5.8}$$

The induced maps of (5.5) and (5.8) exist because of the commutative diagrams (1.16) and (1.15). The vertical maps

$$\begin{array}{c} \bigwedge^b G^* \\ \downarrow \\ \bigwedge^{b+1} G^* \end{array} \tag{5.9}$$

in $(\mathbf{b}^i, d)$ are given by exterior multiplication: $d(x) = \xi \wedge x$ for all $x \in \bigwedge^b G^*$. The other two types of vertical maps in $\mathbf{b}^i$ are more complicated. These maps are where $\mathbf{b}^i$ differs significantly from $\mathbf{B}^i$. The vertical maps in $\mathbf{B}^i$ are all Koszul maps, and, except for column i, they are all linear. The columns of $\mathbf{b}^i$ (other than column $i+1$) are comprised of quadratic maps which are not Koszul maps. The bicomplex $\mathbf{b}^0$ is isomorphic to the bicomplex $(**)$ of [11, Theorem 5.1]; however, our description of the vertical maps differs a great deal from the description given by Buchsbaum and Eisenbud. Before describing the other maps in $\mathbf{b}^i$, we define two intermediate maps.

Definition 5.10. *If the notation of (5.1) is adopted, then define maps*

$$\ell\colon S_aF^* \otimes \bigwedge^b F^* \otimes \bigwedge^c G^* \to S_aF^* \otimes \bigwedge^{b+1} F^* \otimes \bigwedge^c G^* \quad by$$
$$\ell(x) = (1 \otimes \upsilon \otimes 1)(x) - (1 \otimes 1 \otimes \xi)\,\mathbf{s}\,\partial_{\Upsilon^*}(x)$$

and define maps

$$m\colon D_aF \otimes \bigwedge^b F^* \otimes \bigwedge^c G^* \to D_aF \otimes \bigwedge^{b+1} F^* \otimes \bigwedge^c G^* \quad by$$
$$\ll x, m(y) \gg = \ll \ell(x), y \gg$$

for all $x \in S_aF^* \otimes \bigwedge^{f-b-1} F^* \otimes \bigwedge^{g-c} G^*$ *and* $y \in D_aF \otimes \bigwedge^b F^* \otimes \bigwedge^c G^*$, *where*

$$\ll, \gg\colon \left(S_aF^* \otimes \bigwedge^{f-b} F^* \otimes \bigwedge^{g-c} G^*\right) \otimes \left(D_aF \otimes \bigwedge^b F^* \otimes \bigwedge^c G^*\right) \to R$$

is the perfect pairing of (1.3).

Remarks 5.11. The following conventions are used in the above definition and throughout the paper.

(a) We write υ as an abbreviation for the map $\upsilon\wedge \_\_\colon \bigwedge^b F^* \to \bigwedge^{b+1} F^*$ which sends y to $\upsilon \wedge y$.

(b) In the definition of ℓ, we simplified the notation by writing $\mathbf{s}$ instead of $\mathbf{s}\otimes 1$ because the meaning is not ambiguous. This convention is also mentioned between (1.16) and (1.17).

We are now able to define the rest of the maps of (5.2). The vertical maps

$$\begin{array}{c} \mathfrak{L}_a^b \otimes \bigwedge^c G^* \\ \downarrow \\ \mathfrak{L}_a^{b+1} \otimes \bigwedge^c G^* \end{array} \quad \text{and} \quad \begin{array}{c} K_a^b \otimes \bigwedge^c G^* \\ \downarrow \\ K_a^{b+1} \otimes \bigwedge^c G^* \end{array}$$

of $\big(\mathbf{b}^i, d\big)$ are given by

$$\begin{aligned}(5.12)\qquad \mathfrak{L}_a^b \otimes \bigwedge^c G^* \xrightarrow{\mathbf{s}} S_{a-1}F^* \otimes \bigwedge^{b+1} F^* \otimes \bigwedge^c G^* \xrightarrow{\ell} S_{a-1}F^* \otimes \bigwedge^{b+2} F^* \otimes \bigwedge^c G^* \\ \xrightarrow{\partial_{\mathrm{id}}\otimes 1} \mathfrak{L}_a^{b+1} \otimes \bigwedge^c G^* \qquad \text{and}\end{aligned}$$

$$\begin{aligned}(5.13)\qquad K_a^b \otimes \bigwedge^c G^* \subseteq D_aF \otimes \bigwedge^b F^* \otimes \bigwedge^c G^* \xrightarrow{m} D_aF \otimes \bigwedge^{b+1} F^* \otimes \bigwedge^c G^* \\ \xrightarrow{\mathbf{t}} D_{a+1}F \otimes \bigwedge^{b+2} F^* \otimes \bigwedge^c G^* \xrightarrow{\delta_{\mathrm{id}}} K_a^{b+1} \otimes \bigwedge^c G^*,\end{aligned}$$

respectively.

The next result may be compared with (0.10) and Proposition 2.12.

Proposition 5.14. *If the data of (5.1) is adopted, then* $\mathfrak{b}^{g-f-1-i} \cong (\mathfrak{b}^i)^*[-g]$.

Proof. A routine calculation using (5.3), (5.4), and (1.9) shows that

$$\mathbf{b}_{a\,b}^{g-f-1-i} \cong \big(\mathbf{b}^i_{f-1-a\;g-f+1-b}\big)^*$$

is a module isomorphism for all integers a, b, and i. It follows that

$$(5.15)\qquad \mathfrak{b}_j^{g-f-1-i} \cong \big(\mathfrak{b}^i_{g-j}\big)^*$$

is a module isomorphism for all integers i and j. If $1 \le a$, $0 \le b \le f-1$, and $1 \le c$, then the arguments near (2.14) and (2.16) show that the horizontal maps

$$\mathfrak{L}_a^b \otimes \bigwedge^c G^* \xrightarrow{\partial_{\Upsilon^*}} \mathfrak{L}_{a+1}^b \otimes \bigwedge^{c-1} G^* \quad \text{and} \quad K_a^{f-b-1} \otimes \bigwedge^{g-c+1} G^* \xrightarrow{\delta_{\Upsilon^*}} K_{a-1}^{f-b-1} \otimes \bigwedge^{g-c} G^*$$

and the horizontal maps

$$\bigwedge^{b+c} G^* \xrightarrow{d} \mathfrak{L}_1^b \otimes \bigwedge^{c-1} G^* \quad \text{and} \quad K_0^{f-b-1} \otimes \bigwedge^{g-c+1} G^* \xrightarrow{d} \bigwedge^{g-b-c} G^*$$

are dual to one another. It is obvious that exterior multiplication

$$\begin{array}{ccc} \bigwedge^b G^* & & \bigwedge^{g-b-1} G^* \\ {\scriptstyle \xi\wedge\_}\big\downarrow & \text{is dual to} & \big\downarrow{\scriptstyle \xi\wedge\_} \\ \bigwedge^{b+1} G^* & & \bigwedge^{g-b} G^*. \end{array}$$

To show duality among the other vertical maps, it suffices to show that the diagram

$$\begin{array}{ccc} \mathfrak{L}_a^b \otimes \bigwedge^c G^* & \xrightarrow{\cong} & \left(K_{a-1}^{f-b-1} \otimes \bigwedge^{g-c} G^*\right)^* \\ {\scriptstyle \partial_{\text{id}}\,\ell\mathbf{s}}\big\downarrow & & \big\downarrow{\scriptstyle (\delta_{\text{id}}\,\mathbf{t}m)^*} \\ \mathfrak{L}_a^{b+1} \otimes \bigwedge^c G^* & \xrightarrow{\cong} & \left(K_{a-1}^{f-b-2} \otimes \bigwedge^{g-c} G^*\right)^* \end{array} \tag{5.16}$$

commutes for $1 \le a$ and $0 \le b \le f-2$. The horizontal maps in (5.16) are given, as always, by (1.13) and (1.2). We establish that the diagram commutes by showing that

$$\ll \partial_{\text{id}}\,\ell\mathbf{s}(x), \mathbf{t}(y) \gg = \pm \ll x, \mathbf{t}\,\delta_{\text{id}}\,\mathbf{t}m(y) \gg \tag{5.17}$$

for all $x \in \mathfrak{L}_a^b \otimes \bigwedge^c G^*$ and $y \in K_{a-1}^{f-b-2} \otimes \bigwedge^{g-c} G^*$. According to (1.4) and definition 5.10, the left side of (5.17) is equal to $\pm \ll \mathbf{s}(x), m\,\delta_{\text{id}}\,\mathbf{t}(y) \gg$. On the other hand, the domain of y guarantees that $\delta_{\text{id}}\,\mathbf{t}(y) = y - \mathbf{t}\,\delta_{\text{id}}\,y = y$; hence, the left side of (5.17) is equal to

$$\pm \ll \mathbf{s}(x), m(y) \gg = \pm \ll x, \mathbf{t}m(y) \gg .$$

The homotopy $\mathbf{t}$ satisfies parts (c) and (d) of Proposition 1.6; thus, $\mathbf{t} = \mathbf{t}\,\delta_{\text{id}}\,\mathbf{t}$, and (5.17) has been verified. $\square$

Proposition 5.18. *If the data of (5.1) is adopted, then $\mathfrak{b}^i$ is a complex.*

Proof. It suffices to prove that $\mathbf{b}^i$ is a bicomplex. The arguments of Proposition 2.18 show that each row of $\mathbf{b}_i$ is a complex. The product $\xi \wedge \xi = 0$ in $\bigwedge^2 G^*$; hence the $(i+1)^{\text{st}}$ column of $\mathbf{b}^i$ is also a complex. To show that the other columns of $\mathbf{b}^i$ are complexes, it suffices, because of Proposition 5.14, to show that the composition

$$\begin{array}{c} \mathfrak{L}_a^b \otimes \bigwedge^c G^* \\ {\scriptstyle \partial_{\text{id}}\,\ell\mathbf{s}}\big\downarrow \\ \mathfrak{L}_a^{b+1} \otimes \bigwedge^c G^* \\ {\scriptstyle \partial_{\text{id}}\,\ell\mathbf{s}}\big\downarrow \\ \mathfrak{L}_a^{b+2} \otimes \bigwedge^c G^* \end{array}$$

is zero. In fact, we show that the map

$$\ell\mathbf{s}\,\partial_{\text{id}}\,\ell\mathbf{s}\colon S_aF^* \otimes \bigwedge^b F^* \otimes \bigwedge^c G^* \to S_{a-1}F^* \otimes \bigwedge^{b+3} F^* \otimes \bigwedge^c G^* \quad \text{is the zero map} \tag{5.19}$$

for all a, b and c. We begin by observing that

$$\mathbf{s}\ell\mathbf{s} = 0 \tag{5.20}$$

because $\mathbf{s}\ell\mathbf{s} = \mathbf{s}\,(1 \otimes v \otimes 1)\,\mathbf{s} - \mathbf{s}\,(1 \otimes 1 \otimes \xi)\,\mathbf{s}\,\partial_{\Upsilon^*}\,\mathbf{s}$. The map $\mathbf{s}$ commutes with multiplication by $(1 \otimes 1 \otimes \xi)$; and therefore both terms in $\mathbf{s}\ell\mathbf{s}$ are zero by Proposition 1.6 (b). It follows from (5.20) and Proposition 1.6 (a) that $\ell\mathbf{s}\,\partial_{\mathrm{id}}\,\ell\mathbf{s} = \ell\ell\mathbf{s} =$

$$[(1 \otimes v \otimes 1) - (1 \otimes 1 \otimes \xi)\,\mathbf{s}\,\partial_{\Upsilon^*}]\,[(1 \otimes v \otimes 1) - (1 \otimes 1 \otimes \xi)\,\mathbf{s}\,\partial_{\Upsilon^*}]\,\mathbf{s};$$

thus, $\ell\mathbf{s}\,\partial_{\mathrm{id}}\,\ell\mathbf{s} = A + B + C$ for

$$\begin{aligned} A &= -\,(1 \otimes 1 \otimes \xi)\,(1 \otimes v \otimes 1)\,\mathbf{s}\,\partial_{\Upsilon^*}\,\mathbf{s}, \\ B &= -\,(1 \otimes 1 \otimes \xi)\,\mathbf{s}\,(1 \otimes v \otimes 1)\,\partial_{\Upsilon^*}\,\mathbf{s}, \quad \text{and} \\ C &= \,(1 \otimes 1 \otimes \xi)\,\mathbf{s}\,\partial_{\Upsilon^*}\,(1 \otimes 1 \otimes \xi)\,\mathbf{s}\,\partial_{\Upsilon^*}\,\mathbf{s}. \end{aligned}$$

The Koszul algebra $\left(S_\bullet F^* \otimes \bigwedge^\bullet G^*, \partial_{\Upsilon^*}\right)$ is a differential algebra; hence,

$$\partial_{\Upsilon^*}\,(1 \otimes 1 \otimes \xi) = (v \otimes 1 \otimes 1) - (1 \otimes 1 \otimes \xi)\,\partial_{\Upsilon^*}\,.$$

Since $(1 \otimes 1 \otimes \xi)\,\mathbf{s}\,(1 \otimes 1 \otimes \xi) = 0$, we see that

$C = (1 \otimes 1 \otimes \xi)\,\mathbf{s}\,(v \otimes 1 \otimes 1)\,\mathbf{s}\,\partial_{\Upsilon^*}\,\mathbf{s}$, and $\ell\mathbf{s}\,\partial_{\mathrm{id}}\,\ell\mathbf{s} = (1 \otimes 1 \otimes \xi)\,D\,\partial_{\Upsilon^*}\,\mathbf{s}$, where

$D = -\,(1 \otimes v \otimes 1)\,\mathbf{s} - \mathbf{s}\,(1 \otimes v \otimes 1) + \mathbf{s}\,(v \otimes 1 \otimes 1)\,\mathbf{s}$. On the other hand, if y is any element of $\bigwedge^1 F^*$, then

$$(1 \otimes y \otimes 1)\mathbf{s} = \mathbf{s}(y \otimes 1 \otimes 1)\mathbf{s} - \mathbf{s}(1 \otimes y \otimes 1) \tag{5.21}$$

as maps from $S_aF^* \otimes \bigwedge^b F^* \otimes \bigwedge^c G^*$ to $S_{a-1}F^* \otimes \bigwedge^{b+2} F^* \otimes \bigwedge^c G^*$ for all a, b and c. Indeed, the map on the left side of (5.21) is equal to

$$\begin{aligned} (\mathbf{s}\,\partial_{\mathrm{id}} + \partial_{\mathrm{id}}\,\mathbf{s})(1 \otimes y \otimes 1)\mathbf{s} &= \mathbf{s}\,\partial_{\mathrm{id}}(1 \otimes y \otimes 1)\mathbf{s} = \mathbf{s}\,(\,(y \otimes 1 \otimes 1) - (1 \otimes y \otimes 1)\,\partial_{\mathrm{id}}\,)\,\mathbf{s} \\ &= \mathbf{s}(y \otimes 1 \otimes 1)\mathbf{s} - \mathbf{s}(1 \otimes y \otimes 1)(\mathrm{id} - \mathbf{s}\,\partial_{\mathrm{id}}) = \mathbf{s}(y \otimes 1 \otimes 1)\mathbf{s} - \mathbf{s}(1 \otimes y \otimes 1). \end{aligned}$$

We have now established both (5.21) and (5.19).

We next show that the square

$$\begin{array}{ccc} \mathfrak{L}_a^b \otimes \bigwedge^c G^* & \xrightarrow{\partial_{\Upsilon^*}} & \mathfrak{L}_{a+1}^b \otimes \bigwedge^{c-1} G^* \\ {\scriptstyle \partial_{\mathrm{id}}\,\ell\mathbf{s}}\downarrow & & {\scriptstyle \partial_{\mathrm{id}}\,\ell\mathbf{s}}\downarrow \\ \mathfrak{L}_a^{b+1} \otimes \bigwedge^c G^* & \xrightarrow{\partial_{\Upsilon^*}} & \mathfrak{L}_{a+1}^{b+1} \otimes \bigwedge^{c-1} G^* \end{array}$$

commutes for $a \geq 1$. The map

$$S_{a-1}F^* \otimes \bigwedge^{b+1} F^* \otimes \bigwedge^{c} G^* \xrightarrow{\partial_{\mathrm{id}}} \mathfrak{L}_a^b \otimes \bigwedge^{c} G^*$$

is surjective, so it suffices to prove that

$$\partial_{\mathrm{id}}\,\ell\mathbf{s}\,\partial_{\Upsilon^*}\,\partial_{\mathrm{id}} - \partial_{\Upsilon^*}\,\partial_{\mathrm{id}}\,\ell\mathbf{s}\,\partial_{\mathrm{id}} = 0. \tag{5.22}$$

Once ℓ is replaced by $(1\otimes\upsilon\otimes 1) - (1\otimes 1\otimes\xi)\,\mathbf{s}\,\partial_{\Upsilon^*}$, we see that the left side of (5.22) is equal to $A+B+C+D$ for

$$\begin{aligned} A &= \quad \partial_{\mathrm{id}}\,(1\otimes\upsilon\otimes 1)\,\mathbf{s}\,\partial_{\Upsilon^*}\,\partial_{\mathrm{id}}\\ B &= -\,\partial_{\mathrm{id}}\,(1\otimes 1\otimes\xi)\,\mathbf{s}\,\partial_{\Upsilon^*}\,\mathbf{s}\,\partial_{\Upsilon^*}\,\partial_{\mathrm{id}}\\ C &= -\,\partial_{\Upsilon^*}\,\partial_{\mathrm{id}}\,(1\otimes\upsilon\otimes 1)\,\mathbf{s}\,\partial_{\mathrm{id}}, \qquad \text{and}\\ D &= \quad \partial_{\Upsilon^*}\,\partial_{\mathrm{id}}\,(1\otimes 1\otimes\xi)\,\mathbf{s}\,\partial_{\Upsilon^*}\,\mathbf{s}\,\partial_{\mathrm{id}}\,. \end{aligned}$$

Use the fact that ∂_{id} and ∂_{Υ^*} are differential algebra maps in order to write

$$\begin{array}{llll} A = & (\upsilon\otimes 1\otimes 1)\mathbf{s}\,\partial_{\Upsilon^*}\,\partial_{\mathrm{id}} & -\,(1\otimes\upsilon\otimes 1)\,\partial_{\mathrm{id}}\,\partial_{\Upsilon^*} & \\ C = & -(\upsilon\otimes 1\otimes 1)\,\partial_{\Upsilon^*}\,\mathbf{s}\,\partial_{\mathrm{id}} & +\,(1\otimes\upsilon\otimes 1)\,\partial_{\Upsilon^*}\,\partial_{\mathrm{id}} & \\ D = & (\upsilon\otimes 1\otimes 1)\,\partial_{\mathrm{id}}\,\mathbf{s}\,\partial_{\Upsilon^*}\,\mathbf{s}\,\partial_{\mathrm{id}} & & -\,(1\otimes 1\otimes\xi)\,\partial_{\mathrm{id}}(\partial_{\Upsilon^*}\,\mathbf{s}\,\partial_{\Upsilon^*})\mathbf{s}\,\partial_{\mathrm{id}}\\ B = & & & -\,(1\otimes 1\otimes\xi)\,\partial_{\mathrm{id}}\,\mathbf{s}(\partial_{\Upsilon^*}\,\mathbf{s}\,\partial_{\Upsilon^*})\,\partial_{\mathrm{id}} \end{array}$$

Equation (5.22) is established as soon as we show

$$\mathbf{s}\,\partial_{\Upsilon^*}\,\partial_{\mathrm{id}} = (\partial_{\Upsilon^*}\,\mathbf{s}\,\partial_{\mathrm{id}}) - \partial_{\mathrm{id}}\,\mathbf{s}(\partial_{\Upsilon^*}\,\mathbf{s}\,\partial_{\mathrm{id}}), \quad \text{and} \tag{5.23}$$

$$\partial_{\mathrm{id}}(\partial_{\Upsilon^*}\,\mathbf{s}\,\partial_{\Upsilon^*})\mathbf{s}\,\partial_{\mathrm{id}} + \partial_{\mathrm{id}}\,\mathbf{s}(\partial_{\Upsilon^*}\,\mathbf{s}\,\partial_{\Upsilon^*})\,\partial_{\mathrm{id}} = 0. \tag{5.24}$$

The right side of (5.23) is

$$(1-\partial_{\mathrm{id}}\,\mathbf{s})(\partial_{\Upsilon^*}\,\mathbf{s}\,\partial_{\mathrm{id}}) = \mathbf{s}\,\partial_{\mathrm{id}}(\partial_{\Upsilon^*}\,\mathbf{s}\,\partial_{\mathrm{id}}) = \mathbf{s}\,\partial_{\Upsilon^*}\,\partial_{\mathrm{id}},$$

where the last equality holds because ∂_{id} and ∂_{Υ^*} commute and

$$\partial_{\mathrm{id}}\,\mathbf{s}\,\partial_{\mathrm{id}} = \partial_{\mathrm{id}}\,. \tag{5.25}$$

Equality (5.24) follows from (5.25) because

$$\partial_{\mathrm{id}}(\partial_{\Upsilon^*}\,\mathbf{s}\,\partial_{\Upsilon^*}) = \partial_{\Upsilon^*}\,\partial_{\mathrm{id}}\,\mathbf{s}\,\partial_{\Upsilon^*} = \partial_{\Upsilon^*}(1-\mathbf{s}\,\partial_{\mathrm{id}})\,\partial_{\Upsilon^*} = -\,\partial_{\Upsilon^*}\,\mathbf{s}\,\partial_{\mathrm{id}}\,\partial_{\Upsilon^*} = -(\partial_{\Upsilon^*}\,\mathbf{s}\,\partial_{\Upsilon^*})\,\partial_{\mathrm{id}}\,.$$

We show that the square

$$\begin{tikzcd} \bigwedge^{b+c} G^* \arrow[r, "d"] \arrow[d, "\xi\wedge\_"'] & \mathfrak{L}_1^b \otimes \bigwedge^{c-1} G^* \arrow[d, "\partial_{\mathrm{id}}\,\ell\mathbf{s}"'] \\ \bigwedge^{b+c+1} G^* \arrow[r, "d"] & \mathfrak{L}_1^{b+1} \otimes \bigwedge^{c-1} G^* \end{tikzcd}$$

commutes, for $b\ge 0$, by showing that

$$\begin{tikzcd} \bigwedge^{b+c} G^* \arrow[r, "\Delta"] \arrow[d, "\xi\wedge\_"'] & \bigwedge^{b+1} G^* \otimes \bigwedge^{c-1} G^* \arrow[r, "\bigwedge^{b+1}\Upsilon^*"] & \bigwedge^{b+1} F^* \otimes \bigwedge^{c-1} G^* \arrow[d, "\ell"'] \\ \bigwedge^{b+c+1} G^* \arrow[r, "\Delta"] & \bigwedge^{b+2} G^* \otimes \bigwedge^{c-1} G^* \arrow[r, "\bigwedge^{b+2}\Upsilon^*"] & \bigwedge^{b+2} F^* \otimes \bigwedge^{c-1} G^* \end{tikzcd} \tag{5.26}$$

commutes. The counterclockwise composition is

$$(1 \otimes \upsilon \otimes 1) \left(\bigwedge^{b+1} \Upsilon^*\right) \mathit{\Delta} + (-1)^{b+2} (1 \otimes 1 \otimes \xi) \left(\bigwedge^{b+2} \Upsilon^*\right) \mathit{\Delta}.$$

The clockwise composition is

$$(1 \otimes \upsilon \otimes 1) \left(\bigwedge^{b+1} \Upsilon^*\right) \mathit{\Delta} - (1 \otimes 1 \otimes \xi)\, \mathbf{s}\, \partial_{\Upsilon^*} \left(\bigwedge^{b+1} \Upsilon^*\right) \mathit{\Delta}.$$

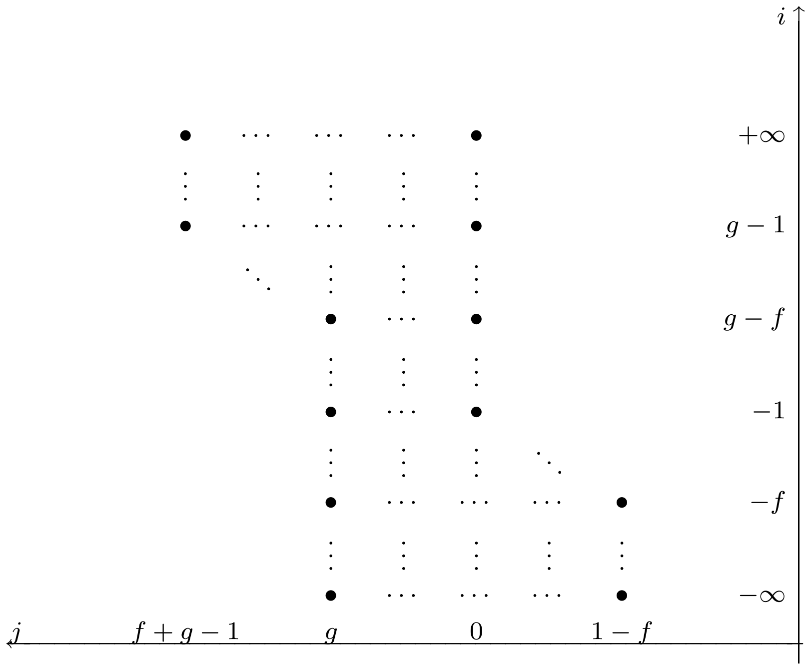
The diagram (5.26) commutes since the co-associative property in $\bigwedge^\bullet G^*$ ensures that

$$\partial_{\Upsilon^*} \left(\bigwedge^{b+1} \Upsilon^*\right) \mathit{\Delta} = (-1)^{b+1}\, \partial_{\mathrm{id}} \left(\bigwedge^{b+2} \Upsilon^*\right) \mathit{\Delta}. \quad \square$$

Section 6. Elementary facts about the complexes $\mathfrak{b}^i$.

In this section we record facts about the complexes $\mathfrak{b}^i$ which are analogous to the results of section 3. The complexes $\mathfrak{b}^i$ are interesting only when $f-1 \le g$ (see Observation 8.10); so no information is lost when we impose this hypothesis.

Observation 6.1. *Adopt the notation of (5.1). If $f-1 \le g$, then $\mathfrak{b}^i_j \neq 0$ if and only if there is a dot at the point (i,j) in the picture below.*



We describe the twists in $\mathfrak{b}^i$ in a manner analogous to Figure 3.4. Each differential map in $\mathbf{b}^i$ may be viewed as a matrix of homogeneous forms. The degrees of

The degrees of the maps in $\mathfrak{b}^i$:

$$\begin{tikzcd}
K_m^{0\,g} \arrow[r, "1"] \arrow[d, "2"'] & . \arrow[r, "1"] & K_0^{0\,f+i+1} \arrow[r, "f"] \arrow[d, "2"'] & \bigwedge^{i+1} \arrow[r, "1"] \arrow[d, "1"'] & \mathfrak{L}_1^{0\,i} \arrow[r, "1"] \arrow[d, "2"'] & . \arrow[r, "1"] & \mathfrak{L}_{i+1}^{0\,0} \arrow[d, "2"'] \\
K_m^{1\,g} \arrow[r, "1"] \arrow[d, "2"'] & . \arrow[r, "1"] & K_0^{1\,f+i+1} \arrow[r, "f-1"] \arrow[d, "2"'] & \bigwedge^{i+2} \arrow[r, "2"] \arrow[d, "1"'] & \mathfrak{L}_1^{1\,i} \arrow[r, "1"] \arrow[d, "2"'] & . \arrow[r, "1"] & \mathfrak{L}_{i+1}^{1\,0} \arrow[d, "2"'] \\
\vdots \arrow[d, "2"'] & & \vdots \arrow[d, "2"'] & \vdots \arrow[d, "1"'] & \vdots \arrow[d, "2"'] & & \vdots \arrow[d, "2"'] \\
K_m^{f-2\,g} \arrow[r, "1"] \arrow[d, "2"'] & . \arrow[r, "1"] & K_0^{f-2\,f+i+1} \arrow[r, "2"] \arrow[d, "2"'] & \bigwedge^{f+i-1} \arrow[r, "f-1"] \arrow[d, "1"'] & \mathfrak{L}_1^{f-2\,i} \arrow[r, "1"] \arrow[d, "2"'] & . \arrow[r, "1"] & \mathfrak{L}_{i+1}^{f-2\,0} \arrow[d, "2"'] \\
K_m^{f-1\,g} \arrow[r, "1"] & . \arrow[r, "1"] & K_0^{f-1\,f+i+1} \arrow[r, "1"] & \bigwedge^{f+i} \arrow[r, "f"] & \mathfrak{L}_1^{f-1\,i} \arrow[r, "1"] & . \arrow[r, "1"] & \mathfrak{L}_{i+1}^{f-1\,0}
\end{tikzcd}$$

Figure 6.2

these forms are recorded in Figure 6.2 where $m = g-f-i-1$, $K_a^{b\,c} = K_a^b \otimes \bigwedge^c G^*$, $\bigwedge^c = \bigwedge^c G^*$, and $\mathfrak{L}_a^{b\,c} = \mathfrak{L}_a^b \otimes \bigwedge^c G^*$.

Notation 6.3. *Retain the data of (5.1). Let $\mathfrak{J}(\Xi, \Upsilon)$ be the ideal $I_1(\Xi\Upsilon) + I_f(\Upsilon)$ of R, $\mathfrak{A}(\Xi, \Upsilon)$ be the ring $R/\mathfrak{J}(\Xi, \Upsilon)$, and $N(\Xi, \Upsilon)$ be the $R-$module*

$$\frac{\operatorname{coker}(\Upsilon^*)}{I_1(\Xi\Upsilon)\operatorname{coker}(\Upsilon^*)}.$$

When there is no ambiguity, we write $\mathfrak{A}$, $\mathfrak{J}$, and N in place of $\mathfrak{A}(\Xi, \Upsilon)$, $\mathfrak{J}(\Xi, \Upsilon)$, and $N(\Xi, \Upsilon)$. If $f \le g$, then $\mathfrak{J}$ is one of the ideals "J" from (0.1); otherwise, $\mathfrak{J}$ is simply equal to $I_1(\Xi\Upsilon)$. It is easy to see that N is actually an $\mathfrak{A}-$module; furthermore, if X and Y are matrices for Ξ and Υ (as described in Observation 3.6 (b)), then there is a surjection from N onto the $\mathfrak{A}-$ideal

$$\frac{I_{f-1}(\text{rows } 1 \text{ to } f-1 \text{ of } Y) + \mathfrak{J}}{\mathfrak{J}}. \tag{6.4}$$

If the data of (5.1) is sufficiently general (see Theorems 9.2 (a) and 9.3 (a), and (10.8)), then the surjection of (6.4) is an isomorphism.

Observation 6.5. *Adopt the notation of (6.3) with $f-1 \le g$. The following statements hold.*

(a)
$$H_0(\mathfrak{b}^i) \cong \begin{cases} \mathfrak{A} & \textit{if } i = 0, \textit{ and} \\ S_i(N) & \textit{if } i \ge 1. \end{cases}$$

(b) *Let Φ be any element of F^*, $\Upsilon_\Phi^* \colon G^* \oplus R \to F^*$ be the extension of Υ^* which sends $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to Φ, and $\mathfrak{J}_\Phi$ be the ideal*

$$\frac{I_1(\Xi\Upsilon) + I_f(\Upsilon_\Phi^*)}{\mathfrak{J}}$$

of $\mathfrak{A}$. Then there is an $R-$module surjection $H_0(\mathfrak{b}^{-1}) \twoheadrightarrow \mathfrak{J}_\Phi$.

NOTE. Recall the notation of Observation 3.6 (b). Let $\Phi_1,\dots,\Phi_f$ be the basis of F^* which is dual to the basis $\varphi_1,\dots,\varphi_f$ of F. If Φ is the element Φ_f of F^*, then $\mathfrak{J}_\Phi$ is the ideal

$$\frac{I_{f-1}(\text{columns } 1 \text{ to } f-1 \text{ of } Y)+\mathfrak{J}}{\mathfrak{J}}. \tag{6.6}$$

If the data of (5.1) is sufficiently general (see Theorems 9.2 (a) and 9.3 (a), and (10.7)), then the surjection of (b) is an isomorphism.

Proof. (a) The assertion holds for $i=0$ because the $R-$modules $\mathfrak{A}$ and $H_0(\mathfrak{b}^0)$ are both presented by

$$[\textstyle\bigwedge^f \Upsilon^* \quad v]: \displaystyle\bigwedge^f G^* \oplus \bigwedge^{f-1} F^* \to \bigwedge^f F^*.$$

We next assume that $i\ge 1$. The modules $S_i(N)$ and $S_i(\operatorname{coker}(\Upsilon^*))\otimes(R/I_1(\Xi\Upsilon))$ are isomorphic, and both modules are presented by

$$[\,\partial_{\Upsilon^*} \quad 1\otimes v\,]: \left(S_{i-1}F^*\otimes \bigwedge^f F^*\otimes G^*\right)\oplus\left(S_iF^*\otimes\bigwedge^{f-1}F^*\right)\to S_iF^*\otimes\bigwedge^f F^*.$$

We observed in (1.12) that $S_iF^*\otimes\bigwedge^{f-1}F^* = \mathbf{s}\mathfrak{L}_{i+1}^{f-2}\oplus\mathfrak{L}_i^{f-1}$. Use (5.2) in order to see that $H_0(\mathfrak{b}^i)$ is presented by

$$[\,\partial_{\Upsilon^*} \quad (1\otimes v)\mathbf{s}\,]: \left(S_{i-1}F^*\otimes \bigwedge^f F^*\otimes G^*\right)\oplus\mathfrak{L}_{i+1}^{f-2}\to S_iF^*\otimes\bigwedge^f F^*.$$

On the other hand, $\mathfrak{L}_i^{f-1}=\partial_{\mathrm{id}}\left(S_{i-1}F^*\otimes\bigwedge^f F^*\right)$; and the fact that $\left(S_\bullet F^*\otimes\bigwedge^\bullet F^*,\partial_{\mathrm{id}}\right)$ is a $DG-$algebra ensures that $(1\otimes v)\,\partial_{\mathrm{id}}\left(S_{i-1}F^*\otimes\bigwedge^f F^*\right)\subseteq \operatorname{im}(\partial_{\Upsilon^*})$.

(b) The module $H_0(\mathfrak{b}^{-1})$ is presented by

$$[\,(\mu\otimes 1)\circ\Upsilon^*\circ(1\otimes \varDelta) \quad \xi\,]: \left(\bigwedge^{f-1}F^*\otimes\bigwedge^f G^*\right)\oplus\bigwedge^{f-2}G^*\to\bigwedge^{f-1}G^*,$$

and the map $\bigwedge^{f-1}G^*\to\mathfrak{J}_\Phi$ which is given by $x\mapsto\left[\Phi\wedge\left(\bigwedge^{f-1}\Upsilon^*\right)(x)\right]$ induces the desired surjection. $\square$

Example 6.7. If $f=1$, then $\mathfrak{b}^i$ is the top row of $\mathbf{b}^i$ for all integers i. Furthermore, in this case, $\mathfrak{b}^i$ is the usual Koszul complex associated to the map $\Upsilon^*\colon G^*\to R$.

Example 6.8. Adopt the notation of (6.3) with $f-1\le g$. If R is local and $I_{f-1}(Y)=R$, then $H_0(\mathfrak{b}^i)=\mathfrak{A}$ for all $i\ge 0$. Indeed, one may choose the bases for F and G so that the matrix of Υ is

$$\left[\begin{array}{c|c} I_{f-1} & 0 \\ \hline & y_{ff} \\ 0 & \vdots \\ & y_{gf} \end{array}\right].$$

It readily follows that $\mathfrak{J} = (x_1, \dots, x_{f-1}, y_{ff}, \dots, y_{gf})$ and $N = R/\mathfrak{J}$.

Example 6.9. In the notation of Example 3.12, let $\mathcal{N} = N(\mathcal{X}, \mathcal{Y})$. If $\operatorname{pd}_R S_i(N) < \infty$, then $\operatorname{pd}_{R[T]} S_i(\mathcal{N}) = \operatorname{pd}_R S_i(N) + 1$.

We conclude this section with a result about the complexes $\mathfrak{b}^i$ which is analogous to Proposition 3.13 and (0.12). In the next result we write $\widetilde{\mathfrak{A}}$, $\widetilde{\mathfrak{b}}$, $\widetilde{\mathbf{b}}$, $\widetilde{\mathfrak{J}}$, and $\widetilde{N}$ to mean $\mathfrak{A}(\widetilde{\Xi}, \widetilde{\Upsilon})$, $\mathfrak{b}(\widetilde{\Xi}, \widetilde{\Upsilon})$, $\mathbf{b}(\widetilde{\Xi}, \widetilde{\Upsilon})$, $\mathfrak{J}(\widetilde{\Xi}, \widetilde{\Upsilon})$, and $N(\widetilde{\Xi}, \widetilde{\Upsilon})$, respectively. We also write $\overline{\mathfrak{A}}$, $\bar{\mathfrak{b}}$, $\bar{\mathbf{b}}$, $\overline{\mathfrak{J}}$, and $\overline{N}$ to mean $\mathfrak{A}(\overline{\Xi}, \Upsilon)$, $\mathfrak{b}(\overline{\Xi}, \Upsilon)$, $\mathbf{b}(\overline{\Xi}, \Upsilon)$, $\mathfrak{J}(\overline{\Xi}, \Upsilon)$, and $N(\overline{\Xi}, \Upsilon)$ respectively.

Theorem 6.10. *In the notation of (5.1), let $G = \widetilde{G} \oplus R\gamma$ for some free $R-$module $\widetilde{G}$ of rank $g-1$, $\widetilde{\Xi}\colon \widetilde{G} \to R$ be the restriction of Ξ to $\widetilde{G}$, $\widetilde{\Upsilon}\colon F \to \widetilde{G}$ be the composition*

$$F \xrightarrow{\Upsilon} G = \widetilde{G} \oplus R\gamma \xrightarrow{\text{proj}} \widetilde{G},$$

and $\overline{\Xi}\colon G \to R$ be the composition

$$G = \widetilde{G} \oplus R\gamma \xrightarrow{\text{proj}} \widetilde{G} \xrightarrow{\widetilde{\Xi}} R.$$

Then, for every integer i, there is a short exact sequence of complexes

$$0 \to \widetilde{\mathfrak{b}}^i \to \bar{\mathfrak{b}}^i \to \widetilde{\mathfrak{b}}^{i-1}[-1] \to 0; \tag{6.11}$$

in particular, there is a long exact sequence of homology

$$\cdots \to \mathrm{H}_{j+1}(\bar{\mathfrak{b}}^i) \to H_j(\widetilde{\mathfrak{b}}^{i-1}) \to H_j(\widetilde{\mathfrak{b}}^i) \to H_j(\bar{\mathfrak{b}}^i) \to H_{j-1}(\widetilde{\mathfrak{b}}^{i-1}) \to \dots \quad . \tag{6.12}$$

If Γ is the element of G^* with $\Gamma(\gamma) = 1$ and $\Gamma|_{\widetilde{G}} = 0$, then the sequences

$$H_1(\widetilde{\mathfrak{b}}^i) \to H_1(\bar{\mathfrak{b}}^i) \to S_{i-1}(\widetilde{N}) \xrightarrow{\Upsilon^*(\Gamma)} S_i(\widetilde{N}) \to S_i(\overline{N}) \to 0, \quad \textit{for } i \geq 1, \textit{ and} \tag{6.13}$$

$$H_1(\widetilde{\mathfrak{b}}^0) \to H_1(\bar{\mathfrak{b}}^0) \to H_0(\widetilde{\mathfrak{b}}^{-1}) \to \overline{\mathfrak{J}}/\widetilde{\mathfrak{J}} \to 0 \tag{6.14}$$

are exact where $S_0(\widetilde{N})$ is taken to mean $\widetilde{\mathfrak{A}}$.

Proof. If we identify $\widetilde{G}^*$ with the submodule of G^* which annihilates γ, then the decomposition

$$G^* = \widetilde{G}^* \oplus R\Gamma \tag{6.15}$$

is dual to the decomposition $G = \widetilde{G} \oplus R\gamma$; moreover (6.15) induces a short exact sequence

$$0 \to \bigwedge^c \widetilde{G}^* \to \bigwedge^c G^* \to \bigwedge^{c-1} \widetilde{G}^* \to 0. \tag{6.16}$$

Recall the definition of $\mathbf{b}^i_{a\,b}$ given in (5.3). For each triple of integers (a, b, i), (6.16) gives rise to a short exact sequence of modules

$$0 \to \widetilde{\mathbf{b}}^i_{a\,b} \to \bar{\mathbf{b}}^i_{a\,b} \to \widetilde{\mathbf{b}}^{i-1}_{a\,b-1} \to 0.$$

It is tedious, but not difficult, to verify that these exact sequences induce the short exact sequence (6.11). (It is crucial, when performing these verifications, to observe that

$$\Upsilon^*|_{\widetilde{G}^*} = \widetilde{\Upsilon}^*, \qquad \Xi\Upsilon = \widetilde{\Xi}\widetilde{\Upsilon}, \qquad \text{and} \qquad \Xi|_{\widetilde{G}} = \widetilde{\Xi}.$$

The middle equation is significant. It is not true, in general, that $\Xi\Upsilon = \widetilde{\Xi}\widetilde{\Upsilon}$; furthermore there is no short exact sequence analogous to (6.11) with $\bar{\mathfrak{b}}$ replaced by $\mathfrak{b}$.) Now that (6.11) is exact, it follows that (6.12) is also exact. If $0 \le i$, then the module $H_0(\mathfrak{b}^i)$ has been identified in Observation 6.5. The exactness of (6.14) follows immediately. The exactness of (6.13) is established as soon as one checks that the connecting homomorphism

$$S_{i-1}(\widetilde{N}) \cong H_0(\widetilde{\mathfrak{b}}^{i-1}) \to H_0(\widetilde{\mathfrak{b}}^i) \cong S_i(\widetilde{N})$$

is multiplication by the element $\Upsilon^*(\Gamma)$ of the symmetric algebra $S_\bullet^{\widetilde{\mathfrak{A}}}(\widetilde{N})$. $\square$

Section 7. The acyclicity of $\mathfrak{b}^i$ for $f \le g$ in the generic case.

Notation 7.1. *Let f and g be positive integers, R_0 be a commutative noetherian ring, $X_{1\times g} = (x_i)$ and $Y_{g\times f} = (y_{ij})$ be matrices of indeterminates, and R be the polynomial ring $R_0[X, Y]$. Let $\Upsilon\colon F \to G$ and $\Xi\colon G \to R$ be the maps of free $R-$modules which are given by X and Y. Adopt the conventions of (5.1). Let* $\mathbf{b}$ *mean* $\mathbf{b}(\Xi, \Upsilon)$*,* $\mathfrak{b}$ *mean* $\mathfrak{b}(\Xi, \Upsilon)$*, and C represent* $\operatorname{coker}\Upsilon^*$*. The symbol $\otimes$ means $\otimes_R$.*

We consider the complexes $\mathfrak{b}^i$ with $g < f$ in section 8. The main result of the present section is Theorem 7.36, where we prove that $\mathfrak{b}^i$ is acyclic provided $-1 \le i$ and $f \le g$. Our proof is similar to the proof of Theorem 4.2; however there are three significant differences. First of all, there is no result in the literature which plays the role of Observation 3.10; instead, we must offer our own proof (Theorem 7.24) of the "base case" $f = g$. Most of this section is devoted to proving Theorem 7.24. The second difference between the proof of Theorem 7.36 and the proof of Theorem 4.2 involves the inductive step. In (3.14) we obtain the complex $\mathfrak{B}^i$, constructed with generic data, as the mapping cone of two complexes constructed with less data. However, the mapping cone argument of (6.11) does not yield the complex $\mathfrak{b}^i$ constructed using generic data; instead one variable must be set equal to zero. The third difference pertains to the very statements of the theorems. Theorem 4.2 is valid for all positive integers f and g; but, Theorem 7.36 requires that f and g satisfy an inequality. Indeed, Observation 8.10 shows that $\mathfrak{b}^i$ is not acyclic for $g \le f - 2$.

The lemmas that we use in the beginning of our proof of Theorem 7.24 (the case $f = g$) also hold whenever $g < f$; and therefore, we re-use them in our proof of Theorem 8.3 (the case $g = f - 1$). The outline of our proof of Theorem 7.24 is straightforward, but steps are intricate. We begin by showing (Lemma 7.7) that row a of $\mathbf{b}^i$ (denoted $\mathbf{b}^i_{a*}$) is acyclic for all a. We conclude the argument by showing that the induced complex of homology

$$\mathbb{H}'(\mathbf{b}^i): \qquad 0 \to H_0(\mathbf{b}^i_{f-1\,*}) \to H_0(\mathbf{b}^i_{f-2\,*}) \to \cdots \to H_0(\mathbf{b}^i_{0*}) \tag{7.2}$$

is also acyclic. The key step (Theorem 7.22) in our proof that (7.2) is acyclic is a linkage argument which involves a Huneke-Ulrich deviation two Gorenstein ideal.

In the course of proving Lemma 7.7, we prove that the homology $H_0(\mathbf{b}^i_{a*})$ is isomorphic to a module of cycles from a complex we have called $\mathbb{G}^b$. For each $b \ge 0$, let $\mathbb{G}^b$ be the following complex of $R-$modules:

$$\begin{aligned}(7.3)\quad \mathbb{G}^b: \quad 0 \to \bigwedge^b G^* \xrightarrow{\bigwedge^b \Upsilon^*} S_0(C) \otimes \bigwedge^b F^* \xrightarrow{\partial_{\mathrm{id}}} S_1(C) \otimes \bigwedge^{b-1} F^* \xrightarrow{\partial_{\mathrm{id}}} \cdots \\ \xrightarrow{\partial_{\mathrm{id}}} S_{b-1}(C) \otimes \bigwedge^1 F^* \xrightarrow{\partial_{\mathrm{id}}} S_b(C) \otimes \bigwedge^0 F^* \to 0,\end{aligned}$$

where $\partial_{\mathrm{id}}\colon S_a(C) \otimes \bigwedge^c F^* \to S_{a+1}(C) \otimes \bigwedge^{c-1} F^*$ is induced by the Koszul differential

$$\partial_{\mathrm{id}}\colon S_aF^* \otimes \bigwedge^c F^* \to S_{a+1}F^* \otimes \bigwedge^{c-1} F^*.$$

Lemma 7.4. *Adopt the notation of (7.1). If $g \le f$, then the complex $\mathbb{G}^b$ is split exact for all $b \ge 0$.*

Proof. Let $\mathbb{A}$ and $\mathbb{B}$ represent the complexes

$$\begin{array}{llllllll}\mathbb{A}: & 0 \to \bigwedge^f G^* & \xrightarrow{0} & \bigwedge^{f-1} G^* & \xrightarrow{0} \dots \xrightarrow{0} & \bigwedge^1 G^* & \xrightarrow{0} & \bigwedge^0 G^*, \\ \mathbb{B}: & 0 \to S^R_\bullet(C) \otimes \bigwedge^f F^* & \xrightarrow{\partial_{\mathrm{id}}} & S^R_\bullet(C) \otimes \bigwedge^{f-1} F^* & \xrightarrow{\partial_{\mathrm{id}}} \dots \xrightarrow{\partial_{\mathrm{id}}} & S^R_\bullet(C) \otimes \bigwedge^1 F^* & \xrightarrow{\partial_{\mathrm{id}}} & S^R_\bullet(C) \otimes \bigwedge^0 F^*.\end{array}$$

and

Observe that $\bigwedge^\bullet \Upsilon^*\colon \mathbb{A} \to \mathbb{B}$ is a map of complexes, and let $\mathbb{M}$ be the mapping cone of $\bigwedge^\bullet \Upsilon^*$. Since $\mathbb{G}^b$ is a graded strand of $\mathbb{M}$, we complete the proof by showing that $\mathbb{M}$ is split exact; and we do this by proving that $\bigwedge^\bullet \Upsilon^*$ induces an isomorphism on homology.

Let S be the ring $S^R_\bullet(F^*)$ and let $\mathbb{D}$ be the double complex

$$\left(S \otimes \bigwedge^\bullet G^*, \partial_{\Upsilon^*}\right) \otimes_S \left(S \otimes \bigwedge^\bullet F^*, \partial_{\mathrm{id}}\right).$$

(We consider D_{ab} to be the module $S \otimes \bigwedge^a G^* \otimes \bigwedge^b F^*$.) Each row of $\mathbb{D}$ is acyclic because $\left(S \otimes \bigwedge^\bullet F^*, \partial_{\mathrm{id}}\right)$ is the minimal $S-$resolution of R. The image of $\Upsilon^*\colon G^* \to F^*$ in S has grade g by [16, Proposition 21]; therefore,

$$\left(S \otimes \bigwedge^\bullet G^*, \partial_{\Upsilon^*}\right) \text{ is an } S\text{−resolution of } S_\bullet(C), \tag{7.5}$$

and each column of $\mathbb{D}$ is acyclic. The homology of the total complex of $\mathbb{D}$ may be computed using either row homology or column homology; that is, if $\mathbb{D}$ is augmented so that each row and column of

$$\begin{array}{ccccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \to & D_{23} & \to & D_{22} & \to & D_{21} & \to & D_{20} & \to & H'_2 \to 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \to & D_{13} & \to & D_{12} & \to & D_{11} & \to & D_{10} & \to & H'_1 \to 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \to & D_{03} & \to & D_{02} & \to & D_{01} & \to & D_{00} & \to & H'_0 \to 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \to & H''_3 & \to & H''_2 & \to & H''_1 & \to & H''_0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & 0 & &
\end{array},$$

which contains some D_{ij}, is exact, then

$$H_\bullet(\mathbb{H}') \cong H_\bullet(\operatorname{Tot}(\mathbb{D})) \cong H_\bullet(\mathbb{H}''), \tag{7.6}$$

where $\mathbb{H}''$ is the row of **column** homology

$$\cdots \to \mathrm{H}''_3 \to \mathrm{H}''_2 \to \mathrm{H}''_1 \to \mathrm{H}''_0,$$

and $\mathbb{H}'$ is the column of **row** homology

$$\cdots \to \mathrm{H}'_3 \to \mathrm{H}'_2 \to \mathrm{H}'_1 \to \mathrm{H}'_0\,.$$

In our situation, $\mathbb{H}' = \mathbb{A}$ and $\mathbb{H}'' = \mathbb{B}$. A straightforward calculation shows that the isomorphism $H_i(\mathbb{A}) \to H_i(\mathbb{B})$ of (7.6), is induced by $\pm\bigwedge^i \Upsilon^*$. For example, we illustrate the path in $\mathbb{D}$ from $H_2(\mathbb{H}') = \bigwedge^2 G^*$ to $H_2(\mathbb{H}'') = H_2(\mathbb{B})$. If y_1 and y_2 are in G^*, then:

$$\begin{array}{ccccc}
 & & & & 1\otimes(y_1\wedge y_2)\otimes 1 \to y_1\wedge y_2 \\
 & & & & \downarrow \\
 & & \begin{matrix} 1\otimes y_2\otimes\Upsilon^*(y_1) \\ -1\otimes y_1\otimes\Upsilon^*(y_2)\end{matrix} & \to & \begin{matrix}\Upsilon^*(y_1)\otimes y_2\otimes 1 \\ -\Upsilon^*(y_2)\otimes y_1\otimes 1\end{matrix} \\
 & & \downarrow & & \\
-1\otimes 1\otimes \bigwedge^2\Upsilon^*(y_1\wedge y_2) & \to & \begin{matrix}\Upsilon^*(y_2)\otimes 1\otimes\Upsilon^*(y_1) \\ -\Upsilon^*(y_1)\otimes 1\otimes \Upsilon^*(y_2)\end{matrix} & & \\
\downarrow & & & & \\
-1\otimes\bigwedge^2\Upsilon^*(y_1\wedge y_2). & & & &
\end{array}$$

□

Lemma 7.7. *Adopt the notation of (7.1). If $g \le f$ and $i \ge 0$, then each row of the bicomplex $\mathbf{b}^i$ is acyclic.*

Proof. Recall that if $0 \le a \le f-1$, then the a^{th} row of $\mathbf{b}^i$ is the complex

$$(\mathbf{b}^i_{a*}, d):\quad 0 \to \bigwedge^{f+i-a} G^* \to \mathfrak{L}_1^{f-a-1}\otimes\bigwedge^{i} G^* \to \mathfrak{L}_2^{f-a-1}\otimes\bigwedge^{i-1} G^* \to \cdots \to \mathfrak{L}_{i+1}^{f-a-1}\otimes\bigwedge^{0} G^*,$$

where the differential d is described in (5.7) and (5.8). The complex $\mathbb{G}^{f+i-a}$ is defined in (7.3). Let $Z_j(\mathbb{F})$ represent the module of cycles in the complex $\mathbb{F}$ at position j. For each pair of integers (a,i) with $0 \le i$ and $0 \le a \le f-1$, consider the $R-$module homomorphism

$$\beta_a^i \colon \mathfrak{L}_{i+1}^{f-a-1} \otimes \bigwedge^0 G^* \to G_{f-a-1}^{f+i-a} \tag{7.8}$$

which is the composition

$$\mathfrak{L}_{i+1}^{f-a-1} \otimes \bigwedge^0 G^* \xrightarrow{\text{incl}} S_{i+1}F^* \otimes \bigwedge^{f-a-1} F^* \otimes \bigwedge^0 G^* \xrightarrow{\text{nat}} S_{i+1}(C) \otimes \bigwedge^{f-a-1} F^* = G_{f-a-1}^{f+i-a}.$$

It is obvious that the image of β_a^i is contained in $Z_{f-a-1}(\mathbb{G}^{f+i-a})$, and it is not difficult to see that the map of (7.8) induces a map

$$H_0(\mathbf{b}_{a*}^i) \to Z_{f-a-1}(\mathbb{G}^{f+i-a}).$$

(The cases $i \ge 1$ and $i = 0$ must be treated separately, but neither case poses any difficulty.) We prove, by induction on i, that each augmented complex

$$\check{\mathbf{b}}_{a*}^i : \qquad 0 \to \mathbf{b}_{a\,i+1}^i \to \cdots \quad \cdots \to \mathbf{b}_{a1}^i \to \mathbf{b}_{a0}^i \xrightarrow{\beta_a^i} Z_{f-a-1}(\mathbb{G}^{f+i-a}) \to 0 \tag{7.9}$$

is exact.

It is clear that the augmented complex $\check{\mathbf{b}}_{a*}^0$ is exact. Indeed, the complex $\mathbb{G}^{f-a}$ is exact by Lemma 7.4; and therefore, the top row of the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \bigwedge^{f-a} G^* \arrow[r, "\bigwedge^{f-a}\Upsilon^*"] \arrow[d, equal] & S_0(C) \otimes \bigwedge^{f-a} F^* \arrow[r, "\partial_{\mathrm{id}}"] \arrow[d, "\partial_{\mathrm{id}}", "\cong"'] & Z_{f-a-1}(\mathbb{G}^{f-a}) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \bigwedge^{f-a} G^* \arrow[r] & \mathfrak{L}_1^{f-a-1} \otimes \bigwedge^0 G^* \arrow[r, "\beta_a^0"] & Z_{f-a-1}(\mathbb{G}^{f-a}) &
\end{tikzcd}$$

is exact. The bottom row is $\check{\mathbf{b}}_{a*}^0$.

It is also clear that $\check{\mathbf{b}}_{0\,*}^i$ is exact for every $i \ge 1$. Indeed, $\check{\mathbf{b}}_{0\,*}^i$ is isomorphic to

$$\begin{aligned} 0 \to S_0F^* \otimes \bigwedge^f F^* \otimes \bigwedge^i G^* \to S_1F^* \otimes \bigwedge^f F^* \otimes \bigwedge^{i-1} G^* \to \cdots \\ \to S_iF^* \otimes \bigwedge^f F^* \otimes \bigwedge^0 G^* \to S_i(C) \otimes \bigwedge^f F^* \to 0, \end{aligned} \tag{7.10}$$

and we observed in (7.5) that this complex of $R-$modules is exact.

Fix a and i with $1 \le a \le f-1$ and $1 \le i$. Assume, by induction, that $\check{\mathbf{b}}_{a-1\,*}^{i-1}$ is acyclic; we prove that $\check{\mathbf{b}}_{a*}^i$ is also acyclic. We begin by producing an acyclic complex $\mathbb{F}$ with the property that there is a short exact sequence of augmented complexes

$$0 \to \check{\mathbf{b}}_{a-1\,*}^{i-1} \to \check{\mathbb{F}} \to \check{\mathbf{b}}_{a*}^i \to 0. \tag{7.11}$$

The modules in $\mathbb{F}$ are defined by

$$F_j = \begin{cases} S_{i-j}F^* \otimes \bigwedge^{f-a} F^* \otimes \bigwedge^{j} G^*, & \text{if } 0 \le j \le i-1, \\ \left(\bigwedge^{f+i-a} G^*\right) \oplus \left(S_0 F^* \otimes \bigwedge^{f-a} F^* \otimes \bigwedge^{i} G^*\right), & \text{if } j = i, \\ \bigwedge^{f+i-a} G^*, & \text{if } j = i+1, \text{ and} \\ 0, & \text{if } i+2 \le j. \end{cases}$$

The differential $F_j \to F_{j-1}$ in $\mathbb{F}$ is given by

$$\begin{cases} \partial_{\Upsilon^*}, & \text{if } 0 \le j \le i-1, \\ \left[\begin{smallmatrix} 0 & \partial_{\Upsilon^*} \end{smallmatrix}\right], & \text{if } j = i, \text{ and} \\ \left[\begin{smallmatrix} \mathrm{id} \\ 0 \end{smallmatrix}\right], & \text{if } j = i+1. \end{cases}$$

It is clear that $\mathbb{F}$ is an acyclic complex. Indeed, when the module $\bigwedge^{f+i-a} G^*$ is split from $\mathbb{F}$, the resulting complex is isomorphic to the resolution $\mathbf{b}^{i}_{0\,*} \otimes \bigwedge^{f-a} F^*$ of $S_i(C) \otimes \bigwedge^{f-a} F^* = G^{f+i-a}_{f-a}$. (See (7.10).) It follows that the augmented complex

$$\check{\mathbb{F}}: \quad \cdots \to F_1 \to F_0 \to G^{f+i-a}_{f-a} \to 0$$

is exact.

We next describe the maps

$$0 \to \check{\mathbf{b}}^{i-1}_{a-1\,j} \to \check{F}_j \to \check{\mathbf{b}}^{i}_{aj} \to 0. \tag{7.12}$$

If $j = -1$, then (7.12) is the sequence

$$0 \to Z_{f-a}(\mathbb{G}^{f+i-a}) \xrightarrow{\text{incl}} G^{f+i-a}_{f-a} \xrightarrow{\partial_{\mathrm{id}}} Z_{f-a-1}(\mathbb{G}^{f+i-a}) \to 0,$$

and we know from Lemma 7.4 that this sequence is exact. If $0 \le j \le i-1$, then (7.12) is the short exact sequence

$$0 \to \mathfrak{L}^{f-a}_{i-j} \otimes \bigwedge^{j} G^* \xrightarrow{\text{incl}} S_{i-j}F^* \otimes \bigwedge^{f-a} F^* \otimes \bigwedge^{j} G^* \xrightarrow{\partial_{\mathrm{id}}} \mathfrak{L}^{f-a-1}_{i-j+1} \otimes \bigwedge^{j} G^* \to 0$$

of (1.14). If $j = i$, then (7.12) is the short exact sequence

$$0 \to \bigwedge^{f+i-a} G^* \to \left(\bigwedge^{f+i-a} G^*\right) \oplus \left(S_0 F^* \otimes \bigwedge^{f-a} F^* \otimes \bigwedge^{i} G^*\right) \to \mathfrak{L}^{f-a-1}_{1} \otimes \bigwedge^{i} G^* \to 0$$

where the two maps are given by

$$\begin{bmatrix} \mathrm{id} \\ (\bigwedge^{f-a} \Upsilon^*) \circ \Delta \end{bmatrix} \qquad \text{and} \qquad \left[-\partial_{\mathrm{id}} \circ (\bigwedge^{f-a} \Upsilon^*) \circ \Delta \quad \partial_{\mathrm{id}} \right].$$

If $j = i+1$, then (7.12) is the short exact sequence

$$0 \to 0 \to \bigwedge^{f-i+a} G^* \xrightarrow{-\mathrm{id}} \bigwedge^{f-i+a} G^* \to 0.$$

Now that the maps in (7.11) have been defined, it is easy to verify that (7.11) is, indeed, a short exact sequence of complexes. Two of the complexes of (7.11) are exact; therefore, the long exact sequence of homology guarantees that $\check{\mathbf{b}}^i_a$ is also exact. $\square$

Recall the complex $\mathbb{H}'(\mathbf{b}^i)$ from 7.2. Let $\mathbb{K}$ be the complex

$$0 \to \bigwedge^0 F^* \xrightarrow{v} \bigwedge^1 F^* \xrightarrow{v} \cdots \xrightarrow{v} \bigwedge^f F^* \tag{7.13}$$

(with $\bigwedge^f F^*$ in position zero).

Lemma 7.14. *Adopt the notation of 7.1. If $g \le f$ and $0 \le i$, then there is a short exact sequence of complexes*

$$0 \to \mathbb{H}'(\mathbf{b}^i)[-1] \to S_{i+1}(C) \otimes \mathbb{K} \to \mathbb{H}'(\mathbf{b}^{i+1}) \to 0. \tag{7.15}$$

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
\xrightarrow{d} & \mathfrak{L}^0_{i+1} \otimes \bigwedge^0 G^* & \xrightarrow{\beta^i_{f-1}} & S_{i+1}(C) \otimes \bigwedge^0 F^* & \longrightarrow & 0 & \\
 & {\scriptstyle d}\downarrow & & {\scriptstyle -1\otimes v}\downarrow & & {\scriptstyle 1\otimes v}\downarrow & \\
\xrightarrow{d} & \mathfrak{L}^1_{i+1} \otimes \bigwedge^0 G^* & \xrightarrow{\beta^i_{f-2}} & S_{i+1}(C) \otimes \bigwedge^1 F^* & \xrightarrow{\partial_{\mathrm{id}}} & S_{i+2}(C) \otimes \bigwedge^0 F^* & \xrightarrow{\partial_{\mathrm{id}}} \\
 & {\scriptstyle d}\downarrow & & {\scriptstyle -1\otimes v}\downarrow & & {\scriptstyle 1\otimes v}\downarrow & \\
\xrightarrow{d} & \mathfrak{L}^2_{i+1} \otimes \bigwedge^0 G^* & \xrightarrow{\beta^i_{f-3}} & S_{i+1}(C) \otimes \bigwedge^2 F^* & \xrightarrow{\partial_{\mathrm{id}}} & S_{i+2}(C) \otimes \bigwedge^1 F^* & \xrightarrow{\partial_{\mathrm{id}}} \\
 & {\scriptstyle d}\downarrow & & {\scriptstyle -1\otimes v}\downarrow & & {\scriptstyle 1\otimes v}\downarrow & \\
 & \vdots & & \vdots & & \vdots & \\
 & {\scriptstyle d}\downarrow & & {\scriptstyle -1\otimes v}\downarrow & & {\scriptstyle 1\otimes v}\downarrow & \\
\xrightarrow{d} & \mathfrak{L}^{f-2}_{i+1} \otimes \bigwedge^0 G^* & \xrightarrow{\beta^i_1} & S_{i+1}(C) \otimes \bigwedge^{f-2} F^* & \xrightarrow{\partial_{\mathrm{id}}} & S_{i+2}(C) \otimes \bigwedge^{f-3} F^* & \xrightarrow{\partial_{\mathrm{id}}} \\
 & {\scriptstyle d}\downarrow & & {\scriptstyle -1\otimes v}\downarrow & & {\scriptstyle 1\otimes v}\downarrow & \\
\xrightarrow{d} & \mathfrak{L}^{f-1}_{i+1} \otimes \bigwedge^0 G^* & \xrightarrow{\beta^i_0} & S_{i+1}(C) \otimes \bigwedge^{f-1} F^* & \xrightarrow{\partial_{\mathrm{id}}} & S_{i+2}(C) \otimes \bigwedge^{f-2} F^* & \xrightarrow{\partial_{\mathrm{id}}} \\
 & \downarrow & & {\scriptstyle -1\otimes v}\downarrow & & {\scriptstyle 1\otimes v}\downarrow & \\
 & 0 & \longrightarrow & S_{i+1}(C) \otimes \bigwedge^f F^* & \xrightarrow{\partial_{\mathrm{id}}} & S_{i+2}(C) \otimes \bigwedge^{f-1} F^* & \xrightarrow{\partial_{\mathrm{id}}} .
\end{array}$$

Diagram 7.16

Proof. Consider Diagram 7.16. The part to the left of the maps β_a^i is the bicomplex $\mathbf{b}^i$; the map β_a^i is defined in (7.8), and the row to the right of β_a^i is a truncation of the complex $\mathbb{G}^{f+i-a}$ from (7.3). Use Lemmas 7.7 and 7.4, together with the exact sequence (7.9), to see that each row of (7.16) is exact. The squares from (7.16) that look like

$$\begin{array}{ccc} S_{i+1}(C)\otimes \bigwedge^p F^* & \xrightarrow{\partial_{\mathrm{id}}} & S_{i+2}(C)\otimes \bigwedge^{p-1} F^* \\ {\scriptstyle -1\otimes\upsilon}\downarrow & & {\scriptstyle 1\otimes\upsilon}\downarrow \\ S_{i+1}(C)\otimes \bigwedge^{p+1} F^* & \xrightarrow{\partial_{\mathrm{id}}} & S_{i+2}(C)\otimes \bigwedge^{p} F^* \end{array}$$

commute because the differential property of the Koszul map ensures that

$$-\partial_{\mathrm{id}}(1\otimes\upsilon) = -(\upsilon\otimes 1) + (1\otimes\upsilon)\,\partial_{\mathrm{id}}; \tag{7.17}$$

furthermore, $\upsilon = \Upsilon^*(\xi)$ is the zero element in $C = \operatorname{coker}\Upsilon^*$. The squares from (7.16) that look like

$$\begin{array}{ccc} \mathfrak{L}_{i+1}^p\otimes \bigwedge^0 G^* & \xrightarrow{\beta} & S_{i+1}(C)\otimes \bigwedge^p F^* \\ {\scriptstyle d}\downarrow & & {\scriptstyle -1\otimes\upsilon}\downarrow \\ \mathfrak{L}_{i+1}^{p+1}\otimes \bigwedge^0 G^* & \xrightarrow{\beta} & S_{i+1}(C)\otimes \bigwedge^{p+1} F^* \end{array} \tag{7.18}$$

also commute. Indeed, (7.18) may be expanded to become

$$\begin{array}{ccccc} \mathfrak{L}_{i+1}^p\otimes \bigwedge^0 G^* & \xrightarrow{\mathrm{incl}} & S_{i+1}F^*\otimes \bigwedge^p F^* & \xrightarrow{\mathrm{nat}} & S_{i+1}(C)\otimes \bigwedge^p F^* \\ {\scriptstyle d}\downarrow & & {\scriptstyle (\upsilon\otimes 1)\mathbf{s}-(1\otimes\upsilon)}\downarrow & & {\scriptstyle -1\otimes\upsilon}\downarrow \\ \mathfrak{L}_{i+1}^{p+1}\otimes \bigwedge^0 G^* & \xrightarrow{\mathrm{incl}} & S_{i+1}F^*\otimes \bigwedge^{p+1} F^* & \xrightarrow{\mathrm{nat}} & S_{i+1}(C)\otimes \bigwedge^{p+1} F^*. \end{array}$$

The square on the right commutes because $\upsilon = 0$ in C. We know, from (5.12), Definition 5.10, and (7.17) that

$$d = \partial_{\mathrm{id}}\,\ell\mathbf{s} = \partial_{\mathrm{id}}\left((1\otimes\upsilon\otimes 1) - (1\otimes 1\otimes\xi)\,\mathbf{s}\,\partial_{\Upsilon^*}\right)\mathbf{s} = \partial_{\mathrm{id}}(1\otimes\upsilon)\mathbf{s} = (\upsilon\otimes 1)\mathbf{s} - (1\otimes\upsilon)\,\partial_{\mathrm{id}}\,\mathbf{s};$$

furthermore, the restriction of $\partial_{\mathrm{id}}\,\mathbf{s}$ to $\mathfrak{L}_{i+1}^p$ is the identity map by Proposition 1.6. We conclude that (7.16) is a commutative diagram with exact rows.

In the course of studying (7.16), we have seen that

$$S_{i+1}(C)\otimes\mathbb{K} \xrightarrow{\partial_{\mathrm{id}}} S_{i+2}(C)\otimes\mathbb{K}[+1] \tag{7.19}$$

is a map of complexes (up to sign). Let $\mathbb{L}^i$ be the kernel of (7.19). Further consideration of (7.16) shows that

$$0 \to \mathbb{L}^i \to S_{i+1}(C)\otimes\mathbb{K} \to \mathbb{L}^{i+1}[+1] \to 0 \tag{7.20}$$

is a short exact sequence of complexes for all $i \ge 0$ and that there is an isomorphism of complexes

$$\mathbb{H}'(\mathbf{b}^i)[-1] \cong \mathbb{L}^i \tag{7.21}$$

for all $i \ge 0$. Combine (7.20) and (7.21) in order to establish (7.15). $\square$

The next result, although crucial to our proof of Theorem 7.24, does not involve any of the complexes discussed in this paper. It is about Huneke-Ulrich deviation two Gorenstein ideals. These ideals were introduced in [18, Proposition 5.9]; they were resolved in [24], and also in [22] and [31].

Theorem 7.22. *Let R_0 be a commutative noetherian ring; $X_{1\times f}$, $Y_{f\times f}$, and $T_{1\times f}$ be matrices of indeterminates; and S be the polynomial ring $R_0[X, Y, T]$. Let $\mathbf{z} = [z_1, \dots, z_f]$ be the product XY, $\boldsymbol{\ell} = [\ell_1, \dots, \ell_f]$ be the product TY^{t}, $\mathbf{a}$ be the sequence $\ell_1, \dots, \ell_f, z_1, \dots, z_{f-1}$, and K be the $S-$ideal $(\mathbf{a}, z_f, \det(Y))$. Then,*

(a) *$\mathbf{a}$ is a regular sequence on S,*
(b) *$((\mathbf{a})\colon z_f) = (\mathbf{a}, T_f)S$, and*
(c) *$((\mathbf{a})\colon T_f) = K$.*

Proof. (a) Let $\overline{Y}$ be the matrix

$$\begin{bmatrix} T_1 & 0 & \cdots & \cdots & 0 \\ X_2 & T_2 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & T_{f-1} & 0 \\ 0 & \cdots & 0 & X_f & T_f \end{bmatrix},$$

and let $\alpha\colon S \to R_0[X, T]$ be the $R_0[X, T]-$algebra map which carries Y to $\overline{Y}$. It suffices to show that $\alpha(\mathbf{a})$ is a regular sequence. It is clear that

$$\alpha(\boldsymbol{\ell}) = (T_1^2, T_2^2 + T_1X_2, \dots, T_f^2 + T_{f-1}X_f)$$

and $\alpha(\mathbf{z}) = (X_2^2 + X_1T_1, X_3^2 + X_2T_2, \dots, X_f^2 + X_{f-1}T_{f-1}, T_fX_f)$. It follows that the radical of $\alpha(\mathbf{a})$ is the ideal $(T_1, \dots, T_f, X_2, \dots, X_f)$.
(c) Let A be the alternating matrix

$$\begin{bmatrix} 0 & Y \\ -Y^{\mathrm{t}} & 0 \end{bmatrix}.$$

Observe that $K = I_1([X \quad T]\,A) + (\operatorname{Pf} A)$. We saw in (a) that $\operatorname{grade} K \ge 2f-1$. It follows that K is a Huneke-Ulrich deviation two Gorenstein ideal of grade $2f-1$. In particular, the theory of linkage tells us that $((\mathbf{a})\colon K)$ is the almost complete intersection $(\mathbf{a}, s)$ for some s in S and that

$$((\mathbf{a})\colon s) = K. \tag{7.23}$$

The theory of linkage actually yield a great deal more information about the element s. Indeed, s appears in the following comparison of minimal resolutions:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & S(-(4f-2)) & \longrightarrow & \cdots & \longrightarrow & S & \longrightarrow & S/(\mathbf{a}) & \longrightarrow & 0 \\ & & {\scriptstyle s}\downarrow & & & & \| & & \downarrow & & \\ 0 & \longrightarrow & S(-(4f-3)) & \longrightarrow & \cdots & \longrightarrow & S & \longrightarrow & S/K & \longrightarrow & 0, \end{array}$$

where the shift $4f-3$ may be found in [24, Theorem 6.1]. It follows that s is a linear form in $S = R_0[T, X, Y]$. (The elements of R_0 have degree zero and the entries of T, X, and Y all have degree one.) Since each element of $\mathbf{a}$ is a homogeneous

form of degree two and T_f is obviously in $((\mathbf{a}) \colon K)$, we conclude that $T_f = s$; and therefore, (7.23) guarantees that $((\mathbf{a}) \colon T_f) = K$.
(b) We know that

$$((\mathbf{a}) \colon z_f) \cap ((\mathbf{a}) \colon \det(Y)) = ((\mathbf{a}) \colon K) = (\mathbf{a}, T_f) \subseteq ((\mathbf{a}) \colon z_f).$$

It suffices to show that $((\mathbf{a}) \colon z_f) \subseteq ((\mathbf{a}) \colon \det(Y))$. Suppose that $s \in ((\mathbf{a}) \colon z_f)$. Write $s = s_0 + s_1$ where $s_0 \in R_0[X, Y]$ and $s_1 \in (T)S$. If we set $T_1 = \dots = T_f = 0$, then the hypothesis $sz_f \in (\mathbf{a})$ implies $s_0 z_f \in (z_1, \dots, z_{f-1})$. We conclude that $s \in (T_1, \dots, T_f, z_1, \dots, z_{f-1})$. It is easy to see that this last ideal is contained in $((\mathbf{a}) \colon \det(Y))$. $\square$

Theorem 7.24. *Adopt the notation of (7.1). If $f = g$ and $-1 \le i$, then $\mathfrak{b}^i$ is acyclic.*

Proof. It suffices to prove that $\mathfrak{b}^i$ is acyclic for $i \ge 0$. (Indeed, Proposition 5.14 shows that $\mathfrak{b}^{-1}$ is the dual of $\mathfrak{b}^0$. On the other hand, $H_0(\mathfrak{b}^0)$ is the perfect $R-$module $\mathfrak{A}$ of Observation 6.5, and the length of $\mathfrak{b}^0$ is equal to $\operatorname{pd}_R H_0(\mathfrak{b}^0)$.) We know from Lemma 7.7 that each row of $\mathbf{b}^i$ is acyclic; thus, it suffices to prove that the complex $\mathbb{H}'(\mathbf{b}^i)$ of (7.2) is acyclic for each $i \ge 0$. We know, from the theory of linkage, that $\mathfrak{b}^0$ and $\mathbb{H}'(\mathbf{b}^0)$ are both acyclic. Fix $i \ge 0$. Assume, by induction, that $\mathbb{H}'(\mathbf{b}^i)$ is acyclic. We prove $\mathbb{H}'(\mathbf{b}^{i+1})$ is also acyclic. The long exact sequence of homology associated to (7.15) is

$$\begin{aligned}(7.25)\quad &\cdots \to H_1(\mathbb{H}'(\mathbf{b}^i)) \to H_2(S_{i+1}(C) \otimes \mathbb{K}) \to H_2(\mathbb{H}'(\mathbf{b}^{i+1})) \to H_0(\mathbb{H}'(\mathbf{b}^i))\\ &\to H_1(S_{i+1}(C) \otimes \mathbb{K}) \to H_1(\mathbb{H}'(\mathbf{b}^{i+1})) \to 0 \to H_0(S_{i+1}(C) \otimes \mathbb{K}) \to H_0(\mathbb{H}'(\mathbf{b}^{i+1})) \to 0.\end{aligned}$$

We complete the proof by showing that $H_j(S_{i+1}(C) \otimes \mathbb{K}) = 0$ whenever $j \ge 2$, and that the map

$$H_0(\mathbb{H}'(\mathbf{b}^i)) \to H_1(S_{i+1}(C) \otimes \mathbb{K}),$$

from (7.25), is an isomorphism. It is easiest to treat all relevant integers i at once; thus, we prove that

$$H_j(S_+(C) \otimes \mathbb{K}) = 0 \quad \text{for } j \ge 2, \tag{7.26}$$

and that the composition

$$S_\bullet F^* \otimes \bigwedge^f F^* \xrightarrow{\partial_{\mathrm{id}}} S_+ F^* \otimes \bigwedge^{f-1} F^* \xrightarrow{\mathrm{nat}} S_+(C) \otimes \bigwedge^{f-1} F^*$$

induces an isomorphism

$$\frac{S_\bullet F^* \otimes \bigwedge^f F^*}{\mathcal{I}} \cong H_1(S_+(C) \otimes \mathbb{K}), \tag{7.27}$$

where $\mathcal{I}$ is the image of the map

$$\left(S_\bullet F^* \otimes \bigwedge^{f-1} F^*\right) \oplus \left(S_\bullet F^* \otimes \bigwedge^f F^* \otimes G^*\right) \oplus \left(S_0 F^* \otimes \bigwedge^f G^*\right) \xrightarrow{D} S_\bullet F^* \otimes \bigwedge^f F^*,$$

with
$$D = \begin{bmatrix} 1\otimes \upsilon & \partial_{\Upsilon^*} & \operatorname{incl}\otimes \bigwedge^f \Upsilon^* \end{bmatrix}.$$
(Our presentation of the module $H_0(\mathbb{H}'(\mathbf{b}^i)) = H_0(\mathfrak{b}^i)$ may be found in the proof of Observation 6.5 (a).)

The assertions of (7.26) and (7.27) are established by appealing to Theorem 7.22. Recall that R is the polynomial ring $R_0[X, Y]$ and that bases for the free $R-$modules were chosen (but not named) in in (7.1). Let $\Phi_1, \dots, \Phi_f$ be the basis for F^*. The $R-$module C is presented by

$$R^f \xrightarrow{Y^{\mathrm{t}}} R^f \to C \to 0.$$

Let S be the ring $S_\bullet F^*$. In other words, S is the polynomial ring $R[T]$, where T is a $1 \times f$ matrix of indeterminates. It follows that $S_\bullet(C)$ is $S/I_1(\boldsymbol{\ell})$ where $\boldsymbol{\ell}$ is the product TY^{t}. The complex $\mathbb{K}$ is the Koszul complex on the entries of $\mathbf{z} = XY$. Theorem 7.22 (a) shows that $z_1, \dots, z_{f-1}$ is a regular sequence on $S_\bullet(C)$; and therefore the properties of Koszul complexes yield that

(7.28) $H_j(S_\bullet(C) \otimes \mathbb{K}) = 0$ for $j \geq 2$, and

(7.29) $H_1(S_\bullet(C) \otimes \mathbb{K}) \xrightarrow{\cong} \frac{(\mathbf{a})\colon z_f}{(\mathbf{a})}$, where $(\mathbf{a}) = I_1(\boldsymbol{\ell}) + (z_1, \dots, z_{f-1})$.

Furthermore, the isomorphism of (7.29) is induced by the map

$$S_\bullet(C) \otimes \bigwedge^{f-1} F^* \longrightarrow \frac{S}{(\mathbf{a})},$$

which sends $\sum s_k(-1)^{k+1}\Phi_1 \wedge \dots \wedge \widehat{\Phi_k} \wedge \dots \wedge \Phi_f$ to the class of s_f. The assertion of (7.26) follows from (7.28). We now turn our attention to (7.27).

It is clear that the element

$$z = \sum_{i=1}^{f} T_k(-1)^{k+1}\Phi_1 \wedge \dots \wedge \widehat{\Phi_k} \wedge \dots \wedge \Phi_f \tag{7.30}$$

of $S_1(C) \otimes \bigwedge^{f-1} F^*$ is a one-cycle in the complex $S_+(C) \otimes \mathbb{K}$. To establish the isomorphism of (7.27), we show that the $S-$module $H_1(S_\bullet(C) \otimes \mathbb{K})$ is generated by the class of z and that the kernel of the $S-$module map

$$S \longrightarrow H_1(S_\bullet(C) \otimes \mathbb{K}), \tag{7.31}$$

which sends 1 to the class of z, is equal to $\mathcal{I}$. However, Theorem 7.22 (b) shows that $((\mathbf{a})\colon z_f) = (\mathbf{a}, T_f)$; and therefore the composition

$$S \longrightarrow H_1(S_\bullet(C) \otimes \mathbb{K}) \xrightarrow{\cong} \frac{(\mathbf{a})\colon z_f}{(\mathbf{a})} \tag{7.32}$$

of (7.31) and (7.29) is a surjection. Furthermore, Theorem 7.22 (c) shows that the kernel of (7.32) is $\mathcal{I}$. The isomorphism of (7.27) has been established and the proof is complete. $\square$

Lemma 7.33. *Adopt the notation of (7.1) with $f \le g$.*

(a) *Let U be the submatrix of Y which consists of rows 1 to $f-1$. If I is the $R-$ideal $I_1(XY)+I_{f-1}(U)+I_f(Y)$, then $g+1 \le \operatorname{grade} I$.*

(b) *If $1 \le t < f$, then $f+g-t+1+(g-f) \le \operatorname{grade}\Big(I_t(Y)+I_1(XY)\Big)$.*

Proof.
(a) Let $\widetilde{X} = [x_1, \ldots, x_{g-1}]$, $\widetilde{Y}$ be the submatrix of Y which consists of rows 1 to $g-1$, and $\widetilde{I}$ be the $R-$ideal $I_1(\widetilde{X}\widetilde{Y})+I_{f-1}(U)+I_f(\widetilde{Y})$. If $f = g$, then $\widetilde{I} = I_1(\widetilde{X}U)+I_{f-1}(U)$, where $\widetilde{X}_{1\times(f-1)}$ and $U_{(f-1)\times f}$ are matrices of indeterminates. In this case, [8, Proposition 4.2] (see also (0.2)) shows that $\widetilde{I}$ has grade $f = g$. If $f < g$, then

$$g \le \operatorname{grade} \widetilde{I} \tag{7.34}$$

by induction on g; and therefore, we may always assume that (7.34) holds. Observe further that

$$g-f+1 \le \operatorname{grade}\Big(I_{f-1}(U)+I_f(\widetilde{Y})\Big). \tag{7.35}$$

(Indeed, if R_0 is a domain, then (7.35) is obvious because $I_f(\widetilde{Y})$ is a perfect prime ideal of R of grade $g-f$ and $I_{f-1}(U) \not\subseteq I_f(\widetilde{Y})$. Moreover, (7.35) is true in full generality; there is nothing to prove unless $f+2 \le g$, and in this case a proof may be obtained by using the notion of generic residual intersection (see [19, Section 3]), because $I_f(\widetilde{Y})$ is a generic $(g-f)-$residual intersection of $I_{f-1}(U)$.)

Fix an arbitrary prime ideal P which contains I. It suffices to show $g+1 \le \operatorname{depth} R_P$. There are two cases. If $x_g \in P$, then $(x_g, \widetilde{I}) \subseteq P$; and therefore, $g+1 \le \operatorname{grade} P$ by (7.34). If $x_g \notin P$, then the localization R_{x_g} is equal to a polynomial ring $R'[y'_{g1}, \ldots, y'_{gf}, \widetilde{Y}]$ for some ring R', where the y'_{gj} are new indeterminates, the entries of $\widetilde{Y}$ are unchanged, and the ideal IR_{x_g} is equal to $(y'_{g1}, \ldots, y'_{gf})+I_{f-1}(U)+I_f(\widetilde{Y})$. The inequality $g+1 \le \operatorname{grade} IR_{x_g}$ follows from (7.35).

(b) The assertion is obvious if $t = 1$. Let P be a prime ideal containing $I_t(Y)+I_1(XY)$. By induction on t, we may assume there is a $(t-1)\times(t-1)$ minor Δ of Y with $\Delta \notin P$. The ring R_Δ is a polynomial ring $R'[X', Y']$ where $X'_{1\times(t-1)}$ and $Y'_{(g-t+1)\times(f-t+1)}$ are matrices of indeterminates and $I_1(X')+I_1(Y') \subseteq PR_\Delta$. It follows that

$$\begin{aligned}
\operatorname{grade} PR_\Delta \ge&(g-t+1)(f-t+1)+(t-1)\\
=&f+g-t+(g-t)(f-t)\\
\ge&f+g-t+(g-f+1)(1)\\
=&f+g-t+1+(g-f). \quad \square
\end{aligned}$$

Theorem 7.36. *Adopt the notation of (7.1). If $f \le g$ and $-1 \le i$, then $\mathfrak{b}^i$ is acyclic and $H_0(\mathfrak{b}^i)$ is isomorphic to an ideal of $H_0(\mathfrak{b}^0)$.*

Proof. Recall the modules $\mathfrak{A} = \mathfrak{A}(X,Y)$, $\mathfrak{J} = \mathfrak{J}(X,Y)$, and $N = N(X,Y)$ from (6.3). The proof proceeds by induction on g. Theorem 7.24 takes care of the acyclicity of $\mathfrak{b}^i$ when $f = g$. We prove that $H_0(\mathfrak{b}^i)$ is isomorphic to an ideal of $\mathfrak{A}$ at the end of the proof. In the mean time, we assume that $f < g$. Take $\widetilde{X}$, $\widetilde{Y}$, and $\widetilde{I}$ as in the proof of Lemma 7.33 (a). Let $\widetilde{\mathfrak{b}}^i = \mathfrak{b}^i(\widetilde{X}, \widetilde{Y})$, $\widetilde{\mathfrak{J}} = I_1(\widetilde{X}\widetilde{Y}) + I_f(\widetilde{Y})$, $\widetilde{\mathfrak{A}} = R/\widetilde{\mathfrak{J}}$, $\widetilde{N} = N(\widetilde{X}, \widetilde{Y})$, and $z = \det(Y')$, where Y' is the $f \times f$ submatrix of Y which consists of rows 1 to $f-1$ and row g. Observe that the image of z in $\widetilde{\mathfrak{A}}$ is equal to $\sum_{j=1}^{f} y_{gj}\Delta_j$, where $\Delta_1, \dots, \Delta_f$ is a generating set for $\widetilde{I}\widetilde{\mathfrak{A}}$. Let $\overline{X}$ be the $1 \times g$ matrix $\begin{bmatrix} \widetilde{X} & 0 \end{bmatrix}$, $\bar{\mathfrak{b}}^i = \mathfrak{b}^i(\overline{X}, Y)$, and $\overline{\mathfrak{J}} = I_1(\overline{X}Y) + I_f(Y)$.

The induction hypothesis guarantees that $\widetilde{\mathfrak{b}}^i$ is acyclic for all $i \ge -1$. Therefore, the long exact sequence (6.12) yields $H_j(\bar{\mathfrak{b}}^i) = 0$ for all i and j with $j \ge 2$ and $i \ge 0$. The following observations are necessary before we consider $H_1(\bar{\mathfrak{b}}^i)$.

(7.37) The $R-$ideal $\widetilde{\mathfrak{J}}$ is perfect of grade $g-1$.

(7.38) The $\widetilde{\mathfrak{A}}-$ideal $\widetilde{I}\widetilde{\mathfrak{A}}$ has positive grade.

(7.39) The element z is regular on $\widetilde{\mathfrak{A}}$.

Buchsbaum and Eisenbud [11, Theorem 5.2] (see also (0.2)) have proved (7.37). Assertion (7.38) follows from (7.37) because Lemma 7.33 shows that $g \le \operatorname{grade} \widetilde{I}$. Hochster's notion of general grade reduction ensures (7.39).

We saw in Observation 6.5 (a), together with (6.4), that there is an $\widetilde{\mathfrak{A}}-$module surjection

$$H_0(\widetilde{\mathfrak{b}}^i) = S_i(\widetilde{N}) \twoheadrightarrow \widetilde{I}^i\,\widetilde{\mathfrak{A}} \tag{7.40}$$

for all $i \ge 0$. Since $\widetilde{I}$ has positive grade (by (7.38)), and $H_0(\widetilde{\mathfrak{b}}^i)$ is isomorphic to an ideal of $\widetilde{\mathfrak{A}}$ (by induction), we conclude that (7.40) is an isomorphism. When the isomorphism of (7.40) is applied to the exact sequence (6.13), the image of "$\Upsilon^*(\Gamma)$" in $\widetilde{I}\widetilde{\mathfrak{A}}$ is z; consequently,

$$0 = H_1(\widetilde{\mathfrak{b}}^i) \to H_1(\bar{\mathfrak{b}}^i) \to \widetilde{I}^{i-1}\,\widetilde{\mathfrak{A}} \xrightarrow{z} \widetilde{I}^i\widetilde{\mathfrak{A}}$$

is exact. Use (7.39) in order to conclude that $H_1(\bar{\mathfrak{b}}^i) = 0$ for all $i \ge 1$.

The ideal $\overline{\mathfrak{J}}$ also contains z; consequently, the same argument as above yields that the surjection $H_0(\widetilde{\mathfrak{b}}^{-1}) \twoheadrightarrow \overline{\mathfrak{J}}/\widetilde{\mathfrak{J}}$ of (6.14) is also an isomorphism. It follows that $H_1(\bar{\mathfrak{b}}^0) = 0$; and therefore, $\bar{\mathfrak{b}}^i$ is acyclic for all $i \ge 0$. The complex $\bar{\mathfrak{b}}^{g-f}$ has length g (see Observation 6.1), and it resolves a prefect $R-$module of projective dimension g; thus, $\bar{\mathfrak{b}}^{-1} \cong \left(\bar{\mathfrak{b}}^{g-f}\right)^* [-g]$ is also acyclic.

View R as a graded ring where each element of R_0 has degree zero and every entry of X and Y has degree one. The short exact sequence

$$0 \to R \xrightarrow{x_g} R \to R/(x_g) \to 0$$

induces a short exact sequence of graded complexes

$$0 \to \mathfrak{b}^i \xrightarrow{x_g} \mathfrak{b}^i \to \bar{\mathfrak{b}}^i \to 0.$$

The corresponding long exact sequence of homology yields that multiplication by x_g is an automorphism of $H_j(\mathfrak{b}^i)$ for all i and j with $1 \le j$ and $-1 \le i$. Since the homology of $\mathfrak{b}^i$ is finitely generated and graded, and x_g has positive degree, we conclude that $\mathfrak{b}^i$ is acyclic for $i \ge -1$.

It remains to show that $H_0(\mathfrak{b}^i)$ is isomorphic to an ideal of $\mathfrak{A}$. Fix $i \ge 1$. It is easy to see that the $\mathfrak{A}-$module $H_0(\mathfrak{b}^i)$ has rank one. Indeed, if P is an associated prime of $\mathfrak{A}$, then $I_{f-1}(Y)\mathfrak{A} \nsubseteq P$ (see (7.38)); hence, Example 6.8 shows that $H_0(\mathfrak{b}^i)_P = \mathfrak{A}_P$. Let j be an integer with $g+1 \le j \le f+g-1$, and let F_j be the radical of the $R-$ideal generated by

$$\{x \in R \mid \operatorname{pd}_{R_x} H_0(\mathfrak{b}^i)_x < j\}. \tag{7.41}$$

A quick look at Example 6.9 shows that $I_1(XY) + I_{f+g-j}(Y) \subseteq F_j$; and therefore, Lemma 7.33 (b) shows that

$$j + 1 + (g - f) \le \operatorname{grade} F_j. \tag{7.42}$$

It follows that $H_0(\mathfrak{b}^i)$ is a torsion-free $\mathfrak{A}-$module. We conclude that the surjection

$$\mathrm{H}_0(\mathfrak{b}^i) \twoheadrightarrow I_{f-1}(U)^i\mathfrak{A} \tag{7.43}$$

is an isomorphism. (The matrix U is defined in the proof of Lemma 7.33.) Finally, we consider the case $i = -1$. We have seen that $H_0(\mathfrak{b}^{g-f})$ is a perfect $R-$module of projective dimension g, and that

$$H_0(\mathfrak{b}^{-1}) = \operatorname{Ext}^g_R(H_0(\mathfrak{b}^{g-f}), R). \tag{7.44}$$

It follows that $H_0(\mathfrak{b}^{-1})$ is a torsion-free $\mathfrak{A}-$module. If $P \in \operatorname{Ass}(\mathfrak{A})$, then Example 6.8 shows that $H_0(\mathfrak{b}^{g-f})_P$ is obtained from R_P by modding out a regular sequence of length g; thus, (7.44) yields that $H_0(\mathfrak{b}^{-1})$ has rank one. Recall, from (6.6), that there is an $\mathfrak{A}-$module surjection

$$H_0(\mathfrak{b}^{-1}) \twoheadrightarrow \frac{I_{f-1}(\text{columns } 1 \text{ to } f-1 \text{ of } Y) + I_1(XY)}{I_1(XY) + I_f(Y)}. \tag{7.45}$$

Since it is easy to see that the $\mathfrak{A}-$ideal on the right side of (7.45) has positive grade, it follows that (7.45) is an isomorphism. $\square$

Section 8. The complexes $\mathfrak{b}^i$ for $g < f$.

Retain the notation of (7.1) and let i be a positive integer. We saw in Theorem 7.36 that $\mathfrak{b}^i$ is acyclic if $f \le g$. In this section we show that $\mathfrak{b}^i$ is acyclic for $g = f-1$ (Theorem 8.3); but $\mathfrak{b}^i$ has non-trivial homology for $g \le f-2$ (Observation 8.10). The proof of Theorem 8.3 is obtained by modifying the proof of Theorem 7.36. We begin with a calculation which is very similar to Theorem 7.22.

Lemma 8.1. *Let R_0 be a commutative noetherian ring; $X_{1\times(f-1)}$, $Y_{(f-1)\times f}$, and $T=[T_1,\dots,T_f]$ be matrices of indeterminates; and S be the polynomial ring $R_0[X,\,Y,\,T]$. Let $\mathbf{z}=[z_1,\dots,z_f]$ be the product XY; $\boldsymbol{\ell}=[\ell_1,\dots,\ell_{f-1}]$ be the product TY^{t}; $\mathbf{a}$ be the sequence $\ell_1,\dots,\ell_{f-1},z_1,\dots,z_{f-1}$; $\widetilde{Y}$ be the submatrix of Y which consists of columns 1 to $f-1$; and K be the $S-$ideal $(\mathbf{a},T_f,\det(\widetilde{Y}))$. Then,*

(a) $\mathbf{a}$ *is a regular sequence on* S,
(b) $((\mathbf{a})\colon z_f)=K$ *and*
(c) $((\mathbf{a})\colon T_f)=(\mathbf{a},z_f)S$.

Proof. (a) Let $\overline{Y}$ be the matrix

$$\begin{bmatrix} X_1 & T_2 & 0 & \cdots & \cdots & 0\\ 0 & X_2 & T_3 & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & X_{f-2} & T_{f-1} & 0\\ 0 & \cdots & \cdots & 0 & X_{f-1} & T_f \end{bmatrix},$$

and let $\alpha\colon S\to R_0[X,\,T]$ be the $R_0[X,\,T]-$algebra map which carries Y to $\overline{Y}$. The assertion holds because the radical of $\alpha(\mathbf{a})$ is the ideal $(X_1,\dots,X_{f-1},T_2,\dots,T_f)$.
(b) Observe that there is a grade $2f-3$ deviation two Huneke-Ulrich Gorenstein ideal $\widetilde{K}$ such that T_f is regular on $R/\widetilde{K}$, and K is the grade $2f-2$ Gorenstein ideal $(\widetilde{K},T_f)$. We know, from the theory of linkage, that there is an element s of S such that

$$(8.2)\qquad (\mathbf{a})\colon K=(\mathbf{a},s)\qquad\text{and}\qquad (\mathbf{a})\colon s=K.$$

Consider S to be a graded polynomial ring under the following grading: all elements of $R_0[y_{1f},\dots,y_{f-1\,f}]$ have degree zero, T_f has degree two, and all other entries of T, X, and $\widetilde{Y}$ have degree one. Observe that z_f is a linear element of S which is also in $(\mathbf{a})\colon K$. Since each a_j is quadratic, we conclude that $s=z_f$; and therefore, (b) follows from (8.2).
(c) We know that

$$((\mathbf{a})\colon T_f)\cap\Big((\mathbf{a})\colon\det(\widetilde{Y})\Big)=((\mathbf{a})\colon K)=(\mathbf{a},z_f)\subseteq((\mathbf{a})\colon T_f).$$

It suffices to show that $((\mathbf{a})\colon T_f)\subseteq\Big((\mathbf{a})\colon\det(\widetilde{Y})\Big)$. Suppose that $s\in((\mathbf{a})\colon T_f)$. Write $s=s_0+s_1$ where $s_0\in R_0[T,\,Y]$ and $s_1\in(X)S$. If we set $X_1=\cdots=X_{f-1}=0$, then the hypothesis $sT_f\in(\mathbf{a})$ implies $s_0T_f\in(\ell_1,\dots,\ell_{f-1})$. Since $\ell_1,\dots,\ell_{f-1},T_f$ is a regular sequence, we conclude that $s\in(X_1,\dots,X_{f-1},\ell_1,\dots,\ell_{f-1})$. It is easy to see that this last ideal is contained in $((\mathbf{a})\colon\det(\widetilde{Y}))$. $\square$

Theorem 8.3. *Adopt the notation of (7.1) with $g=f-1$. If $-1\le i$, then $\mathfrak{b}^i$ is acyclic.*

Proof. The complex $\mathfrak{b}^{-1}$ is the usual Koszul complex on the regular sequence x_1, $\dots,x_{f-1}$; and thus, it is acyclic. The complex $\mathfrak{b}^0$ is obtained from the mapping cone of two Koszul complexes $\bigwedge^\bullet \Upsilon^*\colon \mathbb{K}'[-1] \to \mathbb{K}$ by splitting off the identity map:

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\bigwedge^0 G^* & \xrightarrow{\mathrm{id}} & \bigwedge^0 F^* \\
\xi\downarrow & & \upsilon\downarrow \\
\bigwedge^1 G^* & \xrightarrow{\bigwedge^1 \Upsilon^*} & \bigwedge^1 F^* \\
\xi\downarrow & & \upsilon\downarrow \\
\vdots & & \vdots \\
\xi\downarrow & & \upsilon\downarrow \\
\bigwedge^{f-1} G^* & \xrightarrow{\bigwedge^{f-1} \Upsilon^*} & \bigwedge^{f-1} F^* \\
\downarrow & & \upsilon\downarrow \\
0 & & \bigwedge^f F^*.
\end{array}$$

The complex $\mathbb{K}$ was introduced in (7.13). We introduce the complex $\mathbb{K}'$ at this time; consider $\bigwedge^{f-1} G^*$ to be in position zero. Observe that $H_i(\mathbb{K}) = 0$ for $i \geq 2$, and that $\mathbb{K}'$ is acyclic. One can easily show that $\mathfrak{b}^0_P$ is acyclic for all prime ideals P of R with $\operatorname{grade} P \leq f+1$. Indeed, if $I_1(X) \not\subseteq P$, then the Koszul complex $\mathbb{K}_P$ is acyclic and $\mathbb{K}'_P$ is split exact. If $I_{f-1}(Y) \not\subseteq P$, then $\bigwedge^{f-1} \Upsilon^*$ induces an isomorphism $H_0(\mathbb{K}'_P) \cong H_1(\mathbb{K}_P)$. It follows, from the Acyclicity Lemma, that $\mathfrak{b}^0$ is acyclic.

We proceed as in the proof of Theorem 7.24. Fix $i \geq 0$. Assume, by induction, that $\mathbb{H}'(\mathbf{b}^i)$ is acyclic. We prove that $\mathbb{H}'(\mathbf{b}^{i+1})$ is also acyclic. Lemma 7.14 applies; so, (7.15) and (7.25) are both exact; consequently, it suffices to establish (7.26) and (7.27). Adopt the notation introduced in the paragraph below (7.27); that is, let $S = S_\bullet F^* = R[T]$ for some $1 \times f$ matrix of indeterminates T, $\boldsymbol{\ell}_{1\times(f-1)} = TY^{\mathrm{t}}$, and $\mathbf{z}_{1\times f} = XY$. It follows that $S_\bullet(C) = S/I_1(\boldsymbol{\ell})$. Lemma 8.1 (a) shows that $z_1, \dots, z_{f-1}$ is a regular sequence on $S_\bullet(C)$; thus (7.28) and (7.29) still hold for $(\mathbf{a}) = I_1(\boldsymbol{\ell}) + (z_1, \dots, z_{f-1})$; and therefore, (7.26) also holds.

View S as a graded polynomial ring where each element of R has degree zero and each T_i has degree one. The isomorphism of (7.29) induces

$$H_1(S_+(C) \otimes \mathbb{K}) \cong \left(\frac{(\mathbf{a})\colon z_f}{(\mathbf{a})}\right)_+. \tag{8.4}$$

Lemma 8.1 (b) gives $((\mathbf{a})\colon z_f) = (\mathbf{a}, T_f, \det(\widetilde{Y}))$. Since $(\det(\widetilde{Y}))(T_1, \dots, T_{f-1})$ is contained in $(\mathbf{a}, T_f)$, we conclude that

$$\left(\frac{(\mathbf{a})\colon z_f}{(\mathbf{a})}\right)_+ = \frac{(\mathbf{a}, T_f)}{(\mathbf{a})}. \tag{8.5}$$

In the present context (7.32) becomes

$$S \to H_1(S_+(C) \otimes \mathbb{K}) \xrightarrow{\cong} \left(\frac{(\mathbf{a})\colon z_f}{(\mathbf{a})}\right)_+ \tag{8.6}$$

where the first map still sends 1 to the element z of (7.30) and the second map is the isomorphism of (8.4). Use (8.5) to see that (8.6) is surjective; and use Lemma 8.1 (c) to see that the kernel of (8.6) is the ideal $\mathcal{I}$ of (7.27). $\square$

There are significant differences between the complexes $\mathfrak{b}^i$ of Theorem 8.3 and the complexes $\mathfrak{b}^i$ of Theorem 7.36.

Proposition 8.7. *Adopt the notation of (7.1) with $g = f-1$.*

(a) *The $R-$module $H_0(\mathfrak{b}^0)$ is not perfect.*
(b) *If $i \geq 1$, then the module $H_0(\mathfrak{b}^i)$ is not isomorphic to an ideal of the ring $H_0(\mathfrak{b}^0)$.*

Proof. Let $\mathfrak{A}$ denote $H_0(\mathfrak{b}^0) = R/I_1(XY)$.
(a) The projective dimension of $\mathfrak{A}$ is f, but $\operatorname{grade} I_1(XY) = f-1$.
(b) It is clear that the module $N(X,Y)$ of (6.3) is isomorphic to $I_g(Y) \otimes_R (R/I_1(XY))$. Observation 6.5 shows that

$$H_0(\mathfrak{b}^i) \cong S_i\left(\frac{I_g(Y)}{I_1(XY)I_g(Y)}\right) \tag{8.8}$$

for positive i. Let $\widetilde{Y}$ be the submatrix of Y which consists of rows 2 to $f-1$ and columns 3 to f, and let $x = x_1 \det(\widetilde{Y})$. We will show that

$$\left(\frac{I_g(Y)}{I_1(XY)I_g(Y)}\right)_x \cong \mathfrak{A}_x \oplus \mathfrak{A}_x. \tag{8.9}$$

Indeed, routine row and column operations show that there exist bases for F_x and G_x, a ring $\widetilde{R}$, and indeterminates $y'_1, \dots, y'_f$ such that R_x is the polynomial ring $\widetilde{R}[y'_1, \dots, y'_f]$,

$$\Xi_x = [1 \quad 0 \dots 0] \quad \text{and} \quad \Upsilon_x = \left[\begin{array}{c|c} y'_1 \ y'_2 & y'_3 \ \cdots \ y'_f \\ \hline 0 & I \end{array}\right].$$

It is now clear that $\mathfrak{A}_x = R_x/(y'_1, \dots, y'_f)$ and that the left side of (8.9) is isomorphic to

$$\frac{(y'_1, y'_2)}{(y'_1, \dots, y'_f)(y'_1, y'_2)},$$

which is certainly isomorphic to the right side of (8.9). $\square$

Observation 8.10. *Adopt the notation of (7.1) with $g \leq f-2$. If $0 \leq i$, then $H_1(\mathfrak{b}^i) \neq 0$.*

Proof. The beginning of $\mathfrak{b}^0$ is isomorphic to the beginning of the Koszul complex on the entries of XY:

$$\mathfrak{b}^0: \qquad \cdots \to \bigwedge^{f-2} G^* \oplus \bigwedge^{f-3} F^* \to \bigwedge^{f-2} F^* \xrightarrow{v} \bigwedge^{f-1} F^* \xrightarrow{v} \bigwedge^{f} F^*.$$

Since the entries of XY do not form a regular sequence, it is clear that $\mathfrak{b}^0$ has non-zero homology at $\bigwedge^{f-1} F^*$.

Fix the integer $i \geq 1$. Let $\Phi_1, \dots, \Phi_f$ be a basis for F^* (as was used in the proof of Theorem 7.24),

$$\Delta_j = (-1)^{j+1} \det(\text{the } g \times g \text{ submatrix of Y consisting of columns } 1, \dots, \widehat{j}, \dots, g+1),$$

for $1 \leq j \leq g+1$, and

$$u = (\Phi_f)^i \otimes \sum_{j=1}^{g+1} (-1)^{j+1} \Delta_j \Phi_1 \wedge \dots \wedge \widehat{\Phi_j} \wedge \dots \wedge \Phi_f \quad \in \quad S_i F^* \otimes \bigwedge^{f-1} F^* \otimes \bigwedge^{0} G^*.$$

Observe that

$$(8.11) \qquad (1 \otimes v \otimes 1)\, u = 0 \in S_i F^* \otimes \bigwedge^{f} F^* \otimes \bigwedge^{0} G^*,$$

because

$$XY \begin{bmatrix} \Delta_1 \\ \vdots \\ \Delta_{g+1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} = X0 = 0.$$

The element $\partial_{\mathrm{id}}\, u$ is in $\mathfrak{L}_{i+1}^{f-2} \otimes \bigwedge^0 G^*$. We will prove that

(8.12) there is an element $u' \in \mathfrak{L}_i^{f-1} \otimes \bigwedge^1 G^*$ such that $\partial_{\mathrm{id}}\, u + u' \in Z_1(\mathfrak{b}^i)$; but,
(8.13) $\partial_{\mathrm{id}}\, u + u'$ is not a boundary in $\mathfrak{b}^i$ for any $u' \in \mathfrak{L}_i^{f-1} \otimes \bigwedge^1 G^*$.

We know, from (5.12), that $d(\partial_{\mathrm{id}}\, u) = \partial_{\mathrm{id}}\,(1 \otimes v \otimes 1)\, \mathbf{s}\, \partial_{\mathrm{id}}\, u \in \mathfrak{L}_{i+1}^{f-1} \otimes \bigwedge^0 G^*$. On the other hand, Proposition 1.6 tells us that $\mathbf{s}\, \partial_{\mathrm{id}}\, u = u - \partial_{\mathrm{id}}\, \mathbf{s} u$; thus, we see from (8.11) that $d\,(\partial_{\mathrm{id}}\, u) = \partial_{\mathrm{id}}\,(1 \otimes v \otimes 1)\, \partial_{\mathrm{id}}(\mathbf{s}u)$. The last sentence in the proof of Observation 6.5 (a) finishes the proof of (8.12).

We establish (8.13) by showing that $\partial_{\mathrm{id}}\, u$ is not in the image of

$$\begin{array}{ccc} & & \mathfrak{L}_{i+1}^{f-3} \otimes \bigwedge^0 G^* \\ & & \Big\downarrow{\scriptstyle d} \\ \mathfrak{L}_i^{f-2} \otimes \bigwedge^1 G^* & \xrightarrow{d} & \mathfrak{L}_{i+1}^{f-2} \otimes \bigwedge^0 G^*; \end{array}$$

indeed, we show that $\mathbf{s}\, \partial_{\mathrm{id}}\, u$ is not in the image of

$$\begin{array}{ccc} & & S_i F^* \otimes \bigwedge^{f-2} F^* \otimes \bigwedge^0 G^* \\ & & \Big\downarrow{\scriptstyle 1 \otimes v \otimes 1} \\ S_{i-1} F^* \otimes \bigwedge^{f-1} F^* \otimes \bigwedge^1 G^* & \xrightarrow{\partial_{\Upsilon^*}} & S_i F^* \otimes \bigwedge^{f-1} F^* \otimes \bigwedge^0 G^*. \end{array}$$

Since $\mathbf{s}\,\partial_{\text{id}}\,u = u - \partial_{\text{id}}\,\mathbf{s}u$, it suffices to show that $u \neq (1 \otimes v \otimes 1)\,A + \partial_{\Upsilon^*}\,B + \partial_{\text{id}}\,C$ for any

$$A \in S_iF^* \otimes \bigwedge^{f-2} F^* \otimes \bigwedge^{0} G^*, B \in S_{i-1}F^* \otimes \bigwedge^{f-1} F^* \otimes \bigwedge^{1} G^*, \text{and } C \in S_{i-1}F^* \otimes \bigwedge^{f} F^* \otimes \bigwedge^{0} G^*.$$

Observe that $(1 \otimes v \otimes 1)\,A \in I_1(X)\left(S_iF^* \otimes \bigwedge^{f-1} F^* \otimes \bigwedge^0 G^*\right)$, the basis vector

$$\mathbf{\Phi} = (\Phi_f)^i \otimes \Phi_2 \wedge \ldots \wedge \Phi_f$$

does not appear in $\partial_{\text{id}}\,C$, and the coefficient of $\mathbf{\Phi}$ in $\partial_{\Upsilon^*}\,B$ is an element of the ideal $(y_{1f}, \ldots, y_{gf})$. On the other hand, the coefficient of $\mathbf{\Phi}$ in u is Δ_1. The proof is complete because $\Delta_1 \notin I_1(X) + (y_{1f}, \ldots, y_{gf})$. $\square$

Section 9. Summary of the generic case.

In this section we collect everything which is known about the complexes $\mathfrak{B}^i$ and $\mathfrak{b}^i$ in the generic case. In Theorem 9.2 we consider the cases which pertain to divisors on varieties of complexes. Our main contribution to this theory is assertion (a). The other assertions either follow from (a) or have been proved by DeConcini and Strickland, or Bruns, or Huneke and Ulrich.

Data 9.1. *Let f and g be positive integers, R_0 be a commutative noetherian ring, $X_{1\times g}$ and $Y_{g\times f}$ be matrices of indeterminates, and R be the polynomial ring $R_0[X, Y]$. View*

$$F \xrightarrow{Y} G \qquad \text{and} \qquad G \xrightarrow{X} R$$

as maps of free $R-$modules. Form complexes $\mathfrak{B}^i = \mathfrak{B}^i(X, Y)$ and $\mathfrak{b}^i = \mathfrak{b}^i(X, Y)$ as described in sections 2 and 5. Let

$$\boldsymbol{\mathcal{B}} = \begin{cases} \mathfrak{b}, & \text{if } f < g, \text{ and} \\ \mathfrak{B}, & \text{if } g \leq f, \end{cases} \qquad \text{and} \qquad N = \begin{cases} g - f - 1, & \text{if } f < g, \text{ and} \\ f - g + 1, & \text{if } g \leq f. \end{cases}$$

Let J be the $R-$ideal defined in (0.1) and I' be the $(R/J)-$ideal defined in (0.4). Let $m_0 = \min\{f, g\}, m_1 = \max\{f, g\}$. Let Y'' be the

$$\begin{cases} g \times (f-1) \text{ submatrix of } Y \text{ consisting of columns } 1 \text{ to } f-1, & \text{if } f < g \\ (g-1) \times f \text{ submatrix of } Y \text{ consisting of rows } \quad 1 \text{ to } g-1, & \text{if } g \leq f; \end{cases}$$

and let $\mathfrak{I}$ and I'' be the $(R/J)-$ideals defined by

$$\mathfrak{I} = \frac{I_{m_0-1}(Y'') + J}{J} \qquad \text{and} \qquad I'' = \begin{cases} \mathfrak{I}, & \text{if } f < g, \text{ and} \\ \mathfrak{I} + (x_g)(R/J), & \text{if } g \leq f. \end{cases}$$

Theorem 9.2. *Adopt the notation of (9.1).*

(a) *If $-1 \leq i$, then the complex $\boldsymbol{\mathcal{B}}^i$ is acyclic, $H_0(\boldsymbol{\mathcal{B}}^i)$ is a torsion-free $(R/J)-$module of rank one, and*

$$H_0(\boldsymbol{\mathcal{B}}^i) \cong \begin{cases} I'', & \text{if } i = -1, \\ R/J, & \text{if } i = 0, \text{ and} \\ S_i(I'), & \text{if } 1 \leq i. \end{cases}$$

(b) *If i is any integer, then* $\boldsymbol{\mathcal{B}}^i \cong \left(\boldsymbol{\mathcal{B}}^{N-i}\right)^* [-m_1]$.
(c) *The $R-$module $H_0(\boldsymbol{\mathcal{B}}^i)$ is perfect if and only if $-1 \le i \le N+1$. In the case that $H_0(\boldsymbol{\mathcal{B}}^i)$ is perfect, then it has projective dimension m_1.*
(d) *If R_0 is a Gorenstein ring, then $H_0(\boldsymbol{\mathcal{B}}^N)$ is the canonical module of R/J.*
(e) *The natural map from the i^{th} symmetric power of I' to the ordinary i^{th} power of I' is an isomorphism for all $i \ge 1$.*
(f) *If R_0 is a domain, then J, I', and I'' are all prime ideals of R.*
(g) *If $2 \le m_0$ and R_0 is a normal domain, then the following statements also hold.*
  (i) The ring R/J is also a normal domain.
  (ii) The $(R/J)-$ideals I'' and $(I')^i$ are divisorial for all $i \ge 1$.
  (iii) The symbolic and ordinary i^{th} powers of I' are equal for all $i \ge 1$.
  (iv) The inclusion map $R_0 \to R/J$ induces an isomorphism

$$\mathrm{C}\ell\,(R/J) \cong \mathrm{C}\ell\,(R_0) \oplus \mathbb{Z}.$$

  (v) The summand $\mathbb{Z}$ in $\mathrm{C}\ell\,(R/J)$ is generated by $[I']$.
  (vi) The equation $[I'] + [I''] = 0$ holds in $\mathrm{C}\ell\,(R/J)$.
  (vii) Let M be a reflexive $(R/J)-$module of rank one with $[M] = i[I']$ in $\mathrm{C}\ell\,(R/J)$ for some integer i. Assume that, either R_0 is a Gorenstein ring, or else, R_0 is a Cohen-Macaulay ring and $-1 \le i$. Then M is a Cohen-Macaulay $(R/J)-$module if and only if $-1 \le i \le N+1$.

NOTE. Theorem 9.2 shows that the complexes $\{\boldsymbol{\mathcal{B}}^i\}$ satisfy properties (0.6) – (0.11). If $g \le f$, then the parameters N, s, and ρ have the same meaning as they have for the complexes $\mathfrak{C}^i$ and $\mathfrak{D}^i$. If $f < g$, then these parameters were defined in (0.14) in such a way that properties (0.6) – (0.11) continue to hold.

Proof. (a) In Theorems 4.2 and 7.36 we saw that $\boldsymbol{\mathcal{B}}^i$ is acyclic and that $H_0(\boldsymbol{\mathcal{B}}^i)$ is isomorphic to an ideal of R/J. Surjections

$$H_0(\boldsymbol{\mathcal{B}}^{-1}) \twoheadrightarrow I'' \qquad \text{and} \qquad H_0(\boldsymbol{\mathcal{B}}^i) \twoheadrightarrow S_i(I') \twoheadrightarrow (I')^i$$

are produced in Observations 3.7 and 6.5. The ideals I' and I'' have positive grade; hence all of these surjections are isomorphisms.
(b) The duality of the family $\{\boldsymbol{\mathcal{B}}^i\}$ is established in Propositions 2.12 and 5.14.
(c) The module $H_0(\boldsymbol{\mathcal{B}}^i)$ is torsion-free over R/J. Since J is a perfect $R-$ideal of grade m_1, it follows that the annihilator of the $R-$module $H_0(\boldsymbol{\mathcal{B}}^i)$ has grade m_1. Every map in the resolution $\boldsymbol{\mathcal{B}}^i$ is homogeneous of positive degree (see Figures 3.4 and 6.2); thus, the length of $\boldsymbol{\mathcal{B}}^i$ (which may be found in Observations 3.1 and 6.1) is the projective dimension of $H_0(\boldsymbol{\mathcal{B}}^i)$.
(d) This assertion is an immediate consequence of (b).
(e) The proof of (e) is contained in the proof of (a).
(f) DeConcini and Strickland [12] used Hodge Algebra techniques to prove that J is a prime ideal. An independent proof (based only on the fact that J is perfect of grade m_1) can be formulated along the lines of the proof of [9, Theorem 2.10]. Bruns [7, Lemma 2.3] has proved that I' and I'' are prime ideals.

(g) Assertion (i) is proved in [12]. To prove (ii) and (iii) it suffices to show that the $(R/J)-$modules $(I')^i$ satisfy the Serre condition (S_2). To this end we observe that

$$j+2 \le \operatorname{grade} F_j \quad \text{for} \quad m_1 \le j \le f+g-1,$$

where F_j is the ideal of (4.7) or (7.41). Indeed, the grade of F_j is estimated in (4.8) for $g \le f$, and in (7.42) for $f < g$. Assertions (iv), (v), and (vi) are all proved in [7, Theorem 3.1]. Assume that R_0 is a Cohen-Macaulay ring and consider (vii). It is clear that M is a Cohen-Macaulay module if $-1 \le i \le N+1$ and that M is not Cohen-Macaulay if $N+2 \le i$. Now suppose that R_0 is Gorenstein. Huneke and Ulrich have produced an argument (see [20, Theorem 3.5] or [26, Theorem 2.6]) which shows that M is not Cohen-Macaulay if $i \le -2$. □

We conclude this section by summarizing what is known about the complexes $\mathfrak{b}^i$ when $g \le f$ and the complexes $\mathfrak{B}^i$ when $f < g$. These cases contain almost no information about divisors on varieties of complexes.

Theorem 9.3. *Adopt the notation of (9.1).*

(a) *If $g = f$ and $-1 \le i$, then the complex $\mathfrak{b}^i$ is acyclic, $H_0(\mathfrak{b}^i)$ is a torsion-free $(R/J)-$module of rank one, and*

$$H_0(\mathfrak{b}^i) \cong \begin{cases} I', & \text{if } i=-1,\\ R/J, & \text{if } i=0, \text{ and}\\ S_i\left(\frac{I_{f-1}(Y'')+J}{J}\right) \cong (I_{f-1}(Y''))^i\,(R/J), & \text{if } 1\le i.\end{cases}$$

(b) *If $g = f-1$ and $-1 \le i$, then the complex $\mathfrak{b}^i$ is acyclic, and*

$$H_0(\mathfrak{b}^i) \cong \begin{cases} R/I_1(X) \cong \frac{(D)+I_1(XY)}{I_1(XY)}, & \text{if } i=-1,\\ R/I_1(XY), & \text{if } i=0, \text{ and}\\ S_i\left(\frac{I_g(Y)}{I_1(XY)I_g(Y)}\right), & \text{if } 1\le i,\end{cases}$$

where D is the determinant of the submatrix of Y which consists of columns 1 to $f-1$.

(c) *If $g \le f-2$ and $0 \le i$, then $H_1(\mathfrak{b}^i) \ne 0$.*

(d) *If $f \le g-1$ and $\min\{f-g+1,-1\} \le i$, then the complex $\mathfrak{B}^i$ is acyclic, $H_0(\mathfrak{B}^i)$ is a torsion-free $(R/I_1(XY))-$module of rank one, and*

$$H_0(\mathfrak{B}^i) \cong \begin{cases} \frac{(x_g,\Delta)+I_1(XY)}{I_1(XY)}, & \text{if } i=-1 \text{ and } f=g-1,\\ R/I_1(XY), & \text{if } i=0, \text{ or if } f-g+1\le i\le -1, \text{ and}\\ S_i\left(\frac{I_1(X)}{I_1(XY)}\right) \cong (I_1(X))^i\left(\frac{R}{I_1(XY)}\right), & \text{if } 1\le i,\end{cases}$$

where Δ is the determinant of the $(g-1)\times(g-1)$ matrix which consists of rows 1 to $g-1$ of Y in the case that $g-1 = f$.

Proof. The complexes are shown to be acyclic in Theorems 7.36, 8.3, and 4.2, respectively.

(a) The calculation of $H_0(\mathfrak{b}^i)$ for $i \geq 1$ is made in (7.43) and the calculation of $H_0(\mathfrak{b}^{-1})$ is completed in (7.45) .
(b) The map which sends 1 to D induces an isomorphism

$$\frac{R}{I_1(X)} \cong \frac{(D) + I_1(XY)}{I_1(XY)}. \tag{9.4}$$

Use (5.2) to see that $H_0(\mathfrak{b}^{-1})$ is isomorphic to the left side of (9.4). The calculation of $H_0(\mathfrak{b}^i)$ for positive i may be found in (8.8).
(c) See Observation 8.10.
(d) The module $H_0(\mathfrak{B}^{-1})$, for $f = g-1$, may be read from (4.11). A quick look at (2.2) yields that $H_0(\mathfrak{B}^i) \cong R/I_1(XY)$ for $i = 0$ or $f-g+1 \leq i \leq -1$. For $i \geq 1$, the module $H_0(\mathfrak{B}^i)$ is calculated in Observation 3.7, together with Observation 3.6 (c). The fact that the natural map

$$S_i\left(I_1(X)\left(\tfrac{R}{I_1(XY)}\right)\right) \twoheadrightarrow (I_1(X))^i\left(\tfrac{R}{I_1(XY)}\right)$$

is an isomorphism may be read from (4.9). $\square$

Remarks 9.5. (a) We appear to have two descriptions of $H_0(\mathfrak{B}^{-1})$ for $f \leq g-2$ because (4.11) gives

$$H_0(\mathfrak{B}^{-1}) \cong \frac{(x_g) + I_1(XY)}{I_1(XY)}$$

and Theorem 9.3 (d) gives $H_0(\mathfrak{B}^{-1}) \cong R/I_1(XY)$. These isomorphisms are consistent because $(x_g) + I_1(XY)$ is generated by a regular sequence.
(b) Adopt the notation of (9.1) with R_0 a normal domain and $f = g \geq 2$. We know that the complex $\mathfrak{B}^i$ resolves an element of $\mathrm{C}\ell\,(R/J)$ for each $i \geq -1$. It is easy to see that $\mathfrak{B}^0 \cong \mathfrak{b}^0$ and that $\mathfrak{B}^1 \cong \mathfrak{b}^{-1}$. On the other hand, $H_0(\mathfrak{b}^i)$ is not divisorial for any $i \geq 1$. Indeed, we will show that $H_0(\mathfrak{b}^i)_x$ is isomorphic to a height two ideal in $(R/J)_x$, where x is the element x_g of R. It is easy to see that the ring R_x is equal to the polynomial ring $\widetilde{R}[Y'', Z]$, where $\widetilde{R} = R_0[X, x^{-1}]$, $Y''_{(f-1)\times f}$ and $Z_{1\times f}$ are matrices of indeterminates, Y'' is the submatrix of Y consisting of rows 1 to $f-1$, and Z is the product XY. The ideal JR_x is equal to $I_1(Z)R_x$; thus, $(R/J)_x$ is isomorphic to the polynomial ring $\widetilde{R}[Y'']$. Theorem 9.3 (a) shows that

$$H_0(\mathfrak{b}^i)_x \cong (I_{f-1}(Y''))^i\,(R/J)_x \cong (I_{f-1}(Y''))^i\,\widetilde{R}[Y''].$$

It is clear that the $\widetilde{R}[Y'']-$ideal $I_{f-1}(Y'')$ has height two.

Suppose that R_0 is a normal domain in the situation of Theorem 9.3 (d). If $f \leq g-2$, then Hochster's notion of general grade reduction [14] (see also [4, Proposition 6]) shows that the inclusion $R_0 \to R/I_1(XY)$ induces an isomorphism $\mathrm{C}\ell\,(R_0) \cong \mathrm{C}\ell\,(R/I_1(XY))$. However, it is interesting to notice that properties (0.6) – (0.12) all hold for the family $\{\mathfrak{B}^i\}$ when $f = g-1$. Huneke and Ulrich [20] have explained why some of these properties hold for generic residual intersections (i.e. $g \leq f$); however, these properties hold for the families $\{\mathfrak{B}^i\}$, $\{\mathfrak{C}^i\}$, and $\{\mathfrak{D}^i\}$ even when f is one smaller than the least f for which the notion "$f-$residual intersection" is defined.

Proposition 9.6. *Adopt the notation of (4.1) with R_0 a normal domain and $f = g-1 \geq 2$. Let $A = R/I_1(XY)$, $I' = I_1(X)A$, and $I'' = (x_g, \Delta)A$ for Δ defined in Theorem 9.3 (d). The following statements hold.*

(a) *The ring A is a normal domain.*
(b) *The $A-$ideals $(I')^i$ and I'' are divisorial for all $i \geq 1$.*
(c) *The inclusion $R_0 \to A$ induces an isomorphism $\mathrm{C}\ell\,(A) \cong \mathrm{C}\ell\,(R_0) \oplus \mathbb{Z}$.*
(d) *The summand $\mathbb{Z}$ in $\mathrm{C}\ell\,(A)$ is generated by $[I']$.*
(e) *The equation $[I'] + [I''] = 0$ holds in $\mathrm{C}\ell\,(A)$.*

Note. The other conclusions of Theorem 9.2 also hold for A provided the notation is adjusted correctly.

Proof. Avramov [4, Proposition 11] has proved (a) and (c).
(d) Avramov's proof shows that $[I_f(Y)A]$ generates the summand $\mathbb{Z}$ in $\mathrm{C}\ell\,(A)$. There is no difficulty showing that $I_f(Y)A \cong I_1(X)A$.
(b) If F_j is the ideal defined in (4.7), then it suffices to show that $\operatorname{grade} F_j \geq j+2$ for $f+1 \leq j \leq f+g-1$. On the other hand, the inequality of line (4.8) applies because $f+1 \leq j$ implies $g \leq j$.
(e) It is clear that $I' \cap I''$ is equal to the principal ideal $(x_g)A$. □

Section 10. The non-generic case.

We have seen (in Theorems 4.2 and 7.36) that the complexes $\mathfrak{B}^i$ and $\mathfrak{b}^i$ are acyclic when the data (Ξ, Υ) is generic. In the present section we offer some conditions on non-generic data which are sufficient to ensure that the complexes $\mathfrak{B}^i$ and $\mathfrak{b}^i$ remain acyclic. In the final result of the paper (Theorem 10.17) we interpret the complexes $\{\mathfrak{B}^i\}$ in the context of residual intersection. An expanded form of arguments similar to those of this section may be found in [26, Sections 9 and 11].

We first consider those complexes $\mathfrak{b}^i$ which have the same length as $\mathfrak{b}^0$. In this case the principal of the persistence of perfection (see, for example, [9, Theorem 3.5] or [15, Proposition 6.14]) applies.

Proposition 10.1. *Adopt the notation of (6.3) with $1 \leq f \leq g$. Assume that $\mathfrak{J}$ is a proper ideal of R with $\operatorname{grade} \mathfrak{J} \geq g$. If $-1 \leq i \leq g-f$, then $\mathfrak{b}^i$ is acyclic and $H_0(\mathfrak{b}^i)$ is a perfect $R-$module of projective dimension g.* □

In order to treat complexes $\mathfrak{b}^i$ which are longer than $\mathfrak{b}^0$ we must consider the lower order minors of the map Υ.

Definition 10.2. *The data (Ξ, Υ) of (5.1) is called $\mathfrak{b}-$robust if*

(a) $1 \leq f \leq g$,
(b) $\operatorname{grade}(I_1(\Xi\Upsilon) + I_f(\Upsilon)) \geq g$, *and*
(c) $\operatorname{grade}(I_1(\Xi\Upsilon) + I_t(\Upsilon)) \geq f+g+1-t$ *for all t with $1 \leq t \leq f-1$.*

Theorem 10.3. *Adopt the notation of (6.3). If (Ξ, Υ) is $\mathfrak{b}-$robust and $i \geq -1$, then*

(a) *$\mathfrak{b}^i$ is acyclic, and*
(b) *$H_0(\mathfrak{b}^i)$ is a torsion-free $\mathfrak{A}-$module of rank one.*

Proof. If $-1 \le i \le g-f$, then assertion (a) is proved in Proposition 10.1. The torsion-freeness of $H_0(\mathfrak{b}^i)$ now follows readily; see, for example, Observation 1.19. The rank of $H_0(\mathfrak{b}^i)$ may be calculated as in the proof of Theorem 7.36. Henceforth, i is a fixed integer with $g-f+1 \le i$.

We may assume, without loss of generality, that the ring R is local. The result holds for $f=1$ by Example 6.7. The proof continues by induction on f. Lemma 7.33 (b) and Theorem 7.36 show that the hypotheses and conclusions all hold in the generic case. The arbitrary case may be obtained from the generic case by modding out a regular sequence. Consequently, our proof proceeds as follows. Assume that the conclusions (a) and (b) hold for data $(\widetilde{\Xi},\widetilde{\Upsilon})$ over the local ring $\widetilde{R}$ and that z is a regular element of $\widetilde{R}$ with the property that (Ξ,Υ) is $\mathfrak{b}-$robust data over the ring $R=\widetilde{R}/(z)$ where $\Xi=\widetilde{\Xi}\otimes_{\widetilde{R}}1_R$ and $\Upsilon=\widetilde{\Upsilon}\otimes_{\widetilde{R}}1_R$. We prove that the conclusions (a) and (b) hold for the data (Ξ,Υ).

We follow our usual convention and let $\mathfrak{J}$, $\mathfrak{b}$, $\mathfrak{A}$, $\widetilde{\mathfrak{J}}$, $\widetilde{\mathfrak{b}}$, $\widetilde{\mathfrak{A}}$, and $\bar{\mathfrak{b}}$ mean $\mathfrak{J}(\Xi,\Upsilon)$, $\mathfrak{b}(\Xi,\Upsilon)$, $\mathfrak{A}(\Xi,\Upsilon)$, $\mathfrak{J}(\widetilde{\Xi},\widetilde{\Upsilon})$, $\mathfrak{b}(\widetilde{\Xi},\widetilde{\Upsilon})$, $\mathfrak{A}(\widetilde{\Xi},\widetilde{\Upsilon})$, and $\mathfrak{b}(\overline{\Xi},\overline{\Upsilon})$, respectively. We have already observed that $\mathfrak{b}^0=\widetilde{\mathfrak{b}}^0\otimes_{\widetilde{R}}R$ is acyclic. It follows that $\operatorname{Tor}_j^{\widetilde{R}}(\widetilde{\mathfrak{A}},R)=0$ for all $j\ge 1$. In particular, z is regular on $\widetilde{\mathfrak{A}}$. Since $H_0(\widetilde{\mathfrak{b}}^i)$ is a torsion-free $\widetilde{\mathfrak{A}}-$module, we conclude that z is regular on $H_0(\widetilde{\mathfrak{b}}^i)$; and therefore, $\mathfrak{b}^i$ is acyclic.

Now we show that $H_0(\mathfrak{b}^i)$ is a torsion-free $\mathfrak{A}-$module. For each integer j, with

$$g+1\le j\le g+f-1,$$

let F_j be the radical of the $R-$ideal generated by $\{x\in R \mid \operatorname{pd}_{R_x} H_0(\mathfrak{b}^i)_x<j\}$. It suffices to show that $\operatorname{grade} F_j\ge j+1$. Once we establish that

$$I_1(\Xi\Upsilon)+I_{f+g-j}(\Upsilon)\subseteq F_j, \tag{10.4}$$

then the hypothesis that (Ξ,Υ) is $\mathfrak{b}-$robust completes the proof. We know from (6.3) and Observation 6.5 that $I_1(\Xi\Upsilon)$ annihilates $H_0(\mathfrak{b}^i)$; thus $I_1(\Xi\Upsilon)\subseteq F_j$. By mimicking Example 6.9, we are able to show that

$$I_1(\Upsilon)\subseteq F_{f+g-1}. \tag{10.5}$$

Indeed, suppose that the ideal $I_1(\Upsilon)$ is equal to all of the ring R. One may choose bases for F and G so that

$$\Xi=[\,\Xi' \quad x'\,],\qquad \text{and}\qquad \Upsilon=\begin{bmatrix}\Upsilon' & 0\\ 0 & 1\end{bmatrix}$$

for some element x' of R and some matrices $\Xi'_{1\times(g-1)}$ and $\Upsilon'_{(g-1)\times(f-1)}$ with entries in R. Observe that $\mathfrak{J}=(x',\mathfrak{J}(\Xi',\Upsilon'))$. The grade of $\mathfrak{J}$ is positive; thus, there is an $x''\in\mathfrak{J}(\Xi',\Upsilon')$ so that $x=x'+x''$ regular on R. One may easily check that the data $(\overline{\Xi},\overline{\Upsilon})$ is $\mathfrak{b}-$robust, where $\overline{\Xi}=\Xi'\otimes_R 1_{\overline{R}}$ and $\overline{\Upsilon}=\Upsilon'\otimes_R 1_{\overline{R}}$ for $\overline{R}=R/(x)$. The induction hypothesis on f ensures that

$$\operatorname{pd}_{\overline{R}} H_0(\bar{\mathfrak{b}}^i)\le (f-1)+(g-1)-1.$$

On the other hand, x annihilates the $R-$module $H_0(\mathfrak{b}^i)$, and $H_0(\mathfrak{b}^i)$ and $H_0(\bar{\mathfrak{b}}^i)$ are isomorphic as $\overline{R}-$modules. It follows from homological algebra (see, for example, [21, Theorem 3 in Part III]) that

$$\operatorname{pd}_R H_0(\mathfrak{b}^i) = 1 + \operatorname{pd}_{\overline{R}} H_0(\mathfrak{b}^i) \le f+g-2.$$

The inclusion of (10.5) has been established. The technique can be iterated to establish (10.4), and thereby complete the proof that $H_0(\mathfrak{b}^i)$ is a torsion-free $\mathfrak{A}-$module. The calculation of $\operatorname{rank} H_0(\mathfrak{b}^i)$ which is given in the proof of Theorem 7.36 can be modified to work under the present hypotheses. $\square$

Remark 10.6. Continue to assume the hypotheses of Theorem 10.3. Let Y be the matrix of Υ with respect to some bases for F and G. Recall, from Observation 6.5 together with (6.4) and (6.6), that there are $\mathfrak{A}-$module surjections

$$H_0(\mathfrak{b}^{-1}) \twoheadrightarrow I_{f-1}(\text{columns } 1 \text{ to } f-1 \text{ of } Y)\,\mathfrak{A}, \qquad \text{and} \tag{10.7}$$

(10.8)
$$H_0(\mathfrak{b}^i) \twoheadrightarrow S_i\left(I_{f-1}(\text{rows } 1 \text{ to } f-1 \text{ of } Y)\,\mathfrak{A}\right) \twoheadrightarrow \left(I_{f-1}(\text{rows } 1 \text{ to } f-1 \text{ of } Y)\right)^i\mathfrak{A}.$$

If the $R-$ideal $\left(I_{f-1}(\text{columns } 1 \text{ to } f-1 \text{ of } Y)+\mathfrak{J}\right)$ has grade at least $g+1$, then (10.7) is an isomorphism. If $\operatorname{grade}\left(I_{f-1}(\text{rows } 1 \text{ to } f-1 \text{ of } Y)+\mathfrak{J}\right) \ge g+1$, then each surjection in (10.8) is an isomorphism for all $i \ge 1$.

Our discussion of the complexes $\mathfrak{B}^i$ is completely parallel to the above discussion; consequently we will omit most details.

Proposition 10.9. *Adopt the notation of (3.5) with $0 \le g-1 \le f$. Assume that $\mathcal{J}$ is a proper ideal of R with $\operatorname{grade}\mathcal{J} \ge f$. If $-1 \le i \le f-g+2$, then $\mathfrak{B}^i$ is acyclic and $H_0(\mathfrak{B}^i)$ is a perfect $R-$module of projective dimension f.* $\square$

Definition 10.10. *The data (Ξ, Υ) of (2.1) is called $\mathfrak{B}-$robust if*

(a) $0 \le f$,
(b) $1 \le g$,
(c) $\operatorname{grade}(I_1(\Xi\Upsilon)+I_g(\Upsilon)) \ge f$, *and*
(d) $\operatorname{grade}(I_1(\Xi)+I_t(\Upsilon)) \ge f+g+1-t$ *for all t with* $1 \le t \le g-1$.

Remark. If $f \le g-2$, then hypothesis (d) in the definition of $\mathfrak{B}-$robust is equivalent to

$$\operatorname{grade} I_1(\Xi) \ge g \quad \text{and} \quad \operatorname{grade}(I_1(\Xi)+I_t(\Upsilon)) \ge f+g+1-t \quad \text{for } 1 \le t \le f.$$

Theorem 10.11. *Adopt the notation of (3.5). If (Ξ, Υ) is $\mathfrak{B}-$robust and $i \ge -1$, then*

(a) *$\mathfrak{B}^i$ is acyclic, and*
(b) *$H_0(\mathfrak{B}^i)$ is a torsion-free $A-$module of rank one.*

Proof. The boundary cases $f = 0$ and $g = 1$ are treated in Observation 3.10 and Example 3.9, respectively. The most intricate part of the proof involves showing that

$$\operatorname{grade} F_j \geq j + 1 \tag{10.12}$$

for $f+1 \leq j \leq g+f-1$, where F_j is the radical of the $R-$ideal generated by

$$\{x \in R \mid \operatorname{pd}_{R_x} H_0(\mathfrak{B}^i)_x < j\},$$

for a fixed positive integer i. Example 3.11 shows that

$$I_1(\Xi) \subseteq F_j \quad \text{for all } j \text{ with } f+1 \leq j. \tag{10.13}$$

If $f+1 \leq j \leq g-1$, then (10.13) implies (10.12). If $\max\{g, f+1\} \leq j \leq f+g-1$, then an argument based on Example 3.12 shows that $I_{f+g-j}(\Upsilon) \subseteq F_j$; and therefore, the hypothesis that (Ξ, Υ) is $\mathfrak{B}-$robust yields (10.12). □

Remark 10.14. Continue to assume the hypotheses of Theorem 10.11. Let X and Y be matrices which represent Ξ and Υ. Recall, from Observation 3.7 together with Observation 3.6 and (3.8), that there are $A-$module surjections

$$H_0(\mathfrak{B}^{-1}) \twoheadrightarrow \Big((x_g) + I_{g-1}(\text{rows } 1 \text{ to } g-1 \text{ of } Y)\Big) A, \qquad \text{and} \tag{10.15}$$

$$H_0(\mathfrak{B}^{i}) \twoheadrightarrow S_1\Big(I_1(X)A\Big) \twoheadrightarrow \Big(I_1(X)\Big)^i A. \tag{10.16}$$

If $\operatorname{grade}\Big((x_g) + I_{g-1}(\text{rows } 1 \text{ to } g-1 \text{ of } Y) + \mathcal{J}\Big) \geq f+1$, then (10.15) is an isomorphism. If $\operatorname{grade}\Big(I_1(X) + I_g(Y)\Big) \geq f+1$, then each surjection in (10.16) is an isomorphism for all $i \geq 1$.

Remark. There are at least three directions in which one can generalize the above results. We gave conditions on **all** of the ideals $I_t(\Upsilon)$ which guarantee that **all** of the complexes $\mathfrak{B}^i$ and $\mathfrak{b}^i$, for $i \geq -1$, are acyclic. By only looking at some I_t, one can guarantee that some $\mathfrak{B}^i$ and $\mathfrak{b}^i$ are acyclic. Also, if one imposes a slightly weaker hypothesis on the grade of the minors of Υ, then one can prove that the complexes are acyclic without proving that the zeroth homology is torsion-free. Finally, if one imposes a stronger hypothesis on the grade of the minors of Υ, then one can prove that the zeroth homology satisfies the Serre-type condition $(\widetilde{S}_n)$ for some $n \geq 2$. In particular, one can prove that the zeroth homology is reflexive. All of these ideas are carried out in complete detail (in a somewhat different context) in [26, Sections 9 and 10].

We conclude by interpreting the complexes $\{\mathfrak{B}^i\}$ in the context of residual intersections. The relevant definitions may be found in [17] or [19]. Let $\mathbb{K}(Z)$ represent the Koszul complex associated to the map $Z\colon E \to R$; in other words, $\mathbb{K}(Z)$ is the complex

$$\cdots \to \bigwedge^3 E \xrightarrow{\partial_Z} \bigwedge^2 E \xrightarrow{\partial_Z} E \xrightarrow{Z} R.$$

Theorem 10.17. *Let I be a grade g complete intersection ideal in the Cohen-Macaulay local ring $(R,\mathfrak{m},k)$, let $J=(K:I)$ be an $f-$residual intersection, and let t denote the minimal number of generators of I/K. Assume that $1\le t\le g\le f$. If either*

(i) *the ring R is Gorenstein, or else,*
(ii) *the residual intersection $J=(K:I)$ is geometric,*

then there exist matrices $X_{1\times t}$, $Y_{t\times(f-g+t)}$, and $Z_{1\times(g-t)}$ with entries in $\mathfrak{m}$ such that the following statements hold.

(a) *The ideal J is equal to $I_1(Z)+I_1(XY)+I_t(Y)$.*
(b) *The complex $\mathfrak{B}^0(X,Y)\otimes\mathbb{K}(Z)$ is a minimal resolution of R/J by free $R-$modules.*
(c) *If $1\le i\le f-g+2$, then $\mathfrak{B}^i(X,Y)\otimes\mathbb{K}(Z)$ is the minimal $R-$free resolution of $S_i(I/K)$. If, in addition, the residual intersection is geometric, then $S_i(I/K)\cong\bar{I}^i$, where $\bar{I}$ represents the ideal $(I+J)/J$ of R/J.*

Proof. Choose a generating set $z_1,\dots,z_{g-t},x_1,\dots,x_t$ for I with $\bar{z}_1,\dots,\bar{z}_{g-t}$ a basis for $\frac{\mathfrak{m}I+K}{\mathfrak{m}I}$ and $\bar{x}_1,\dots,\bar{x}_t$ is a basis for $\frac{I}{\mathfrak{m}I+K}$. Let Z be the matrix $[z_1,\dots,z_{g-t}]$ and X be the matrix $[x_1,\dots,x_t]$. There exists a matrix $Y_{t\times(f-g+t)}$ with entries in $\mathfrak{m}$, so that K is generated by $I_1(Z)+I_1(XY)$. If hypothesis (ii) is in effect, then (a) is stated as Theorem 4.8 of [8]. The techniques of [26, Section 11] (which are borrowed from [19]) establish (a) under hypothesis (i). The grade of $J'=I_1(XY)+I_t(Y)$ is at most $f-g+t$ (by (0.2)); but J, which has grade f by hypothesis, is equal to $J'+(z_1,\dots,z_{g-t})$. It follows that J' has grade $f-g+t$ and that $z_1,\dots,z_{g-t}$ is a regular sequence on R/J'. Fix an integer i with $-1\le i\le f-g+2$. We may apply Proposition 10.9 and Observation 1.19 in order to conclude that $\mathfrak{B}^i(X,Y)$ is acyclic and that $z_1,\dots,z_{g-t}$ is a regular sequence on $H_0(\mathfrak{B}^i(X,Y))$. It follows that $\mathfrak{B}^i(X,Y)\otimes\mathbb{K}(Z)$ is acyclic. Use Observations 3.7 (a) and 3.6 (c) in order to see that

$$H_0(\mathfrak{B}^i(X,Y))\otimes\tfrac{R}{I_1(Z)}\cong\begin{cases}\frac{R}{J'}\otimes\frac{R}{I_1(Z)}\cong\frac{R}{J}, & \text{if } i=0, \text{ and}\\ S_i\left(\frac{I_1(X)}{I_1(XY)}\right)\otimes\frac{R}{I_1(Z)}\cong S_i\left(\frac{I_1(X)}{I_1(XY)}\otimes\frac{R}{I_1(Z)}\right), & \text{if } i\ge 1.\end{cases}$$

Assertion (b) is established; we now prove (c) . Recall that $z_1,\dots,z_{g-t}$ is a regular sequence on $R/I_1(X)$. It follows that

$$\frac{I_1(X)}{I_1(XY)}\otimes\frac{R}{I_1(Z)}\cong\frac{I}{K}.$$

Finally, we suppose that $J=(K:I)$ is a geometric residual intersection. In this case, $\bar{I}$ has positive grade and $I\cap J=K$. Consider the natural map $S_i(I/K)\twoheadrightarrow\bar{I}^i$. The $(R/J)-$module $S_i(I/K)$ has rank one because I/K is isomorphic to the ideal $\bar{I}$ of R/J; furthermore, Observation 1.19, applied to the $R-$module $H_0(\mathfrak{B}^i(X,Y))$, shows that $S_i(I/K)$ is torsion-free as an $(R/J)-$module. $\square$

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