THE RESOLUTION OF THE UNIVERSAL RING FOR MODULES OF RANK ZERO AND PROJECTIVE DIMENSION TWO

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Abstract. Hochster established the existence of a commutative noetherian ring $\mathcal{R}$ and a universal resolution $\mathbb{U}$ of the form $0 \to \mathcal{R}^e \to \mathcal{R}^f \to \mathcal{R}^g \to 0$ such that for any commutative noetherian ring S and any resolution $\mathbb{V}$ equal to $0 \to S^e \to S^f \to S^g \to 0$, there exists a unique ring homomorphism $\mathcal{R} \to S$ with $\mathbb{V} = \mathbb{U} \otimes_{\mathcal{R}} S$. In the present paper we assume that $f = e + g$ and we find a resolution $\mathbb{F}$ of $\mathcal{R}$ by free $\mathcal{P}$-modules, where $\mathcal{P}$ is a polynomial ring over the ring of integers. The resolution $\mathbb{F}$ is not minimal; but it is straightforward, coordinate free, and independent of characteristic. Furthermore, one can use $\mathbb{F}$ to calculate $\operatorname{Tor}_\bullet^{\mathcal{P}}(\mathcal{R}, \mathbb{Z})$. If e and g both at least 5, then $\operatorname{Tor}_\bullet^{\mathcal{P}}(\mathcal{R}, \mathbb{Z})$ is not a free abelian group; and therefore, the graded betti numbers in the minimal resolution of $\boldsymbol{K} \otimes_{\mathbb{Z}} \mathcal{R}$ by free $\boldsymbol{K} \otimes_{\mathbb{Z}} \mathcal{P}$-modules depend on the characteristic of the field $\boldsymbol{K}$. We record the modules in the minimal $\boldsymbol{K} \otimes_{\mathbb{Z}} \mathcal{P}$ resolution of $\boldsymbol{K} \otimes_{\mathbb{Z}} \mathcal{R}$ in terms of the modules which appear when one resolves divisors over the determinantal ring defined by the 2×2 minors of an $e \times g$ matrix.

Introduction.

Fix positive integers e, f, and g, with $r_1 \geq 1$ and $r_0 \geq 0$, for r_1 and r_0 defined to be $f - e$ and $g - f + e$, respectively. Hochster [12, Theorem 7.2] established the existence of a commutative noetherian ring $\mathcal{R}$ and a universal resolution

$$\mathbb{U}: \quad 0 \to \mathcal{R}^e \to \mathcal{R}^f \to \mathcal{R}^g \to 0$$

such that for any commutative noetherian ring S and any resolution

$$\mathbb{V}: \quad 0 \to S^e \to S^f \to S^g \to 0,$$

there exists a unique ring homomorphism $\mathcal{R} \to S$ with $\mathbb{V} = \mathbb{U} \otimes_{\mathcal{R}} S$. Hochster showed that the universal ring $\mathcal{R}$ is integrally closed and finitely generated as an

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algebra over $\mathbb{Z}$. Huneke [13] identified the generators of $\mathcal{R}$ as a $\mathbb{Z}$-algebra. These generators correspond to the entries of the two matrices from $\mathbb{U}$ and the $\binom{g}{r_1}$ multipliers from the factorization theorem of Buchsbaum and Eisenbud [6, Theorem 3.1]. Bruns [2] showed that $\mathcal{R}$ is factorial. Bruns [3] also showed that universal resolutions exist only for resolutions of length at most two. Heitmann [11] used Bruns' approach to universal resolutions in his counterexample to the rigidity conjecture. Pragacz and Weyman [20] found the relations on the generators of $\mathcal{R}$ and proved that $\boldsymbol{K} \otimes_{\mathbb{Z}} \mathcal{R}$ has rational singularities when $\boldsymbol{K}$ is a field of characteristic zero. Tchernev [21] used the theory of Gröbner bases to generalize and extend all of the above results with special interest in allowing an arbitrary base ring R_0.

When r_1 is equal to 1, then the universal ring $\mathcal{R}$ is the polynomial ring over $\mathbb{Z}$ with variables which represent entries of the second matrix from $\mathbb{U}$ together with variables which represent the Buchsbaum-Eisenbud multipliers. In particular, when $g = r_1 = 1$, then the Hilbert-Burch theorem, which classifies all resolutions of the form

$$0 \to \mathcal{R}^{f-1} \to \mathcal{R}^f \to \mathcal{R}^1 \to 0,$$

is recovered. When $e = 1$ and $r_0 = 0$, then the universal resolution looks like

$$0 \to \mathcal{R}^1 \to \mathcal{R}^f \to \mathcal{R}^{f-1} \to 0,$$

and the universal ring $\mathcal{R}$ is defined by the generic Herzog ideal of grade f [1]. The minimal resolution of $\mathcal{R}$ is given in [18].

The present paper concerns the universal ring $\mathcal{R}$ when $r_0 = 0$. In this case, $f = e + g$ and $\mathcal{R} = \mathcal{P}/\mathcal{J}$, for $\mathcal{P}$ equal to the polynomial ring $\mathbb{Z}[\mathfrak{b}, \{v_{jk}\}, \{x_{ij}\}]$, with $1 \le k \le e$, $1 \le j \le f$, and $1 \le i \le g$, where $\{\mathfrak{b}\} \cup \{v_{jk}\} \cup \{x_{ij}\}$ is a list of indeterminates over $\mathbb{Z}$. We give $\mathcal{J}$ in the language of [21]. Let V be the $f \times e$ matrix and X be the $g \times f$ matrix with entries v_{jk} and x_{ij}, respectively. For each

$$\text{partition of } \{1, \dots, f\} \text{ into } A \cup \bar{A} \text{ with } |A| = e \text{ and } |\bar{A}| = g, \tag{0.1}$$

let $\nabla_{\bar{A},A}$ be the sign of the permutation which arranges the elements of $\bar{A}, A$ into increasing order, $V(A)$ the submatrix of V consisting of the rows from A, and $X(\bar{A})$ the submatrix of X consisting of the columns from $\bar{A}$. In this notation, the ideal which defines the universal ring $\mathcal{R}$ is

$$\mathcal{J} = I_1(XV) + (\{\det X(\bar{A}) + \nabla_{\bar{A},A} \mathfrak{b} \det V(A) \mid A \cup \bar{A} \text{ from } (0.1)\}). \tag{0.2}$$

We produce a resolution $\mathbb{F}$ of $\mathcal{R}$ by free $\mathcal{P}$-modules. Let E, F, and G be free $\mathcal{P}$ modules of rank e, $f = e + g$, and g, respectively; and view the matrices V and X as homomorphisms of $\mathcal{P}$-modules:

$$E \xrightarrow{V} F \xrightarrow{X} G.$$

The resolution $\mathbb{F}$ is given in Definition 2.4. It is infinite, but straightforward, coordinate free, and independent of characteristic. One can view $\mathbb{F}$ as the mapping cone of the following map of complexes:

$$\begin{array}{c} \bigoplus_{(a,c,d)} D_aE \otimes D_cG^* \otimes \bigwedge^d(E \otimes G^*) \otimes \bigwedge^{a-c+e} F^* \\ \big\downarrow \\ \bigwedge^\bullet(E \otimes G^*). \end{array} \tag{0.3}$$

The map $X \circ V \colon E \to G$ induces a map $E \otimes G^* \to R$ which gives rise to an ordinary Koszul complex

$$\cdots \to \bigwedge^d(E \otimes G^*) \to \bigwedge^{d-1}(E \otimes G^*) \to \dots . \tag{0.4}$$

Also, the map $X \colon F \to G$ gives rise to the Koszul complex

$$\cdots \to S_cG \otimes \bigwedge^b F \to S_{c+1}G \otimes \bigwedge^{b-1} F \to \dots$$

and its dual

$$\cdots \to D_{c+1}G^* \otimes \bigwedge^{b-1} F^* \to D_cG^* \otimes \bigwedge^b F^* \to \dots; \tag{0.5}$$

the map $V^* \colon F^* \to E^*$ gives rise to the Koszul complex

$$\cdots \to S_aE^* \otimes \bigwedge^b F^* \to S_{a+1}E^* \otimes \bigwedge^{b-1} F^* \to \dots$$

and its dual

$$\cdots \to D_{a+1}E \otimes \bigwedge^{f-b+1} F^* \to D_aE \otimes \bigwedge^{f-b} F^* \to \dots; \tag{0.6}$$

and the identity map on $E^* \otimes G$ gives rise to the Koszul complex

$$\cdots \to S_aE^* \otimes S_cG \otimes \bigwedge^d(E^* \otimes G) \to S_{a+1}E^* \otimes S_{c+1}G \otimes \bigwedge^{d-1}(E^* \otimes G) \to \dots \tag{0.7}$$

and its dual

$$\cdots \to D_{a+1}E \otimes D_{c+1}G^* \otimes \bigwedge^{d-1}(E \otimes G^*) \to D_aE \otimes D_cG^* \otimes \bigwedge^d(E \otimes G^*) \to \dots . \tag{0.8}$$

The bottom complex of (0.3) is the Koszul complex (0.4). The differential on the top complex of (0.3) involves the maps from (0.4), (0.5), (0.6), and (0.8). The map from the top complex to the bottom complex involves the minors of X and $\mathfrak{b}$ times minors of V. The proof that $\mathbb{F}$ is a resolution of $\mathcal{R}$ uses the acyclicity lemma and induction on e. The case $e=1$ is treated in section 3. In this case, $\mathcal{R}$ is defined by a Herzog ideal and the resolution of $\mathcal{R}$ is already known. In section 5, we exhibit a finite free subcomplex $\mathbb{G}$ of $\mathbb{F}$ which has the same homology as $\mathbb{F}$. We complete the proof that $\mathbb{G}$ and $\mathbb{F}$ are acyclic in section 6. On many occasions we filter a given complex over a partially ordered set; the basic procedure is sumarized in section 4.

All calculations in sections 1 – 6 are made over the ring of integers $\mathbb{Z}$; however, it is necassary to work over a field in section 7. One can use $\mathbb{F}$ to calculate $\operatorname{Tor}_\bullet^{\mathcal{P}}(\mathcal{R},\mathbb{Z})$. If e and g both at least 5 and $\boldsymbol{K}$ is a field, then the Hilbert function of the graded vector space $\operatorname{Tor}_\bullet^{\mathcal{P}\otimes_{\mathbb{Z}}\boldsymbol{K}}(\mathcal{R}\otimes_{\mathbb{Z}}\boldsymbol{K},\boldsymbol{K})$ depends on the characteristic of $\boldsymbol{K}$; and therefore, the graded betti numbers in the minimal resolution of $\boldsymbol{K}\otimes_{\mathbb{Z}}\mathcal{R}$ by free $\boldsymbol{K}\otimes_{\mathbb{Z}}\mathcal{P}$-modules also depend on the characteristic of the field $\boldsymbol{K}$. In particular, $\operatorname{Tor}_\bullet^{\mathcal{P}}(\mathcal{R},\mathbb{Z})$ is not a free abelain group. In Theorem 7.6, we record the modules in the minimal $\boldsymbol{K}\otimes_{\mathbb{Z}}\mathcal{P}$ resolution of $\boldsymbol{K}\otimes_{\mathbb{Z}}\mathcal{R}$ in terms of the modules which appear when one resolves divisors over the determinantal ring defined by the 2×2 minors of an $e\times g$ matrix. Inspired by Theorem 7.6, Kustin and Weyman [19] used the geometric technique for calculating syzygies in order to give a completely different calculation of the $\boldsymbol{K}\otimes_{\mathbb{Z}}\mathcal{P}$ resolution of $\boldsymbol{K}\otimes_{\mathbb{Z}}\mathcal{R}$, when $\boldsymbol{K}$ is a field of characteristic zero. The resolution in [19] is expressed in terms of Schur modules.

1. Preliminary results.

In this paper "ring" means commutative noetherian ring with one. If R is a ring and F is an R-module, then we let F^* denote $\operatorname{Hom}_R(F,R)$.

Let F be a free R-module of finite rank. We make much use of the exterior algebra $\bigwedge^\bullet F$ and the divided power algebra $D_\bullet F$; we make some use of the symmetric algebra $S_\bullet F$ and the tensor algebra $T_\bullet F$. In particular, $\bigwedge^\bullet F$ and $\bigwedge^\bullet F^*$ are modules over one another. Indeed, if $\alpha_i\in\bigwedge^i F^*$ and $b_j\in\bigwedge^j F$, then

$$\alpha_i(b_j)\in\textstyle\bigwedge^{j-i}F \quad\text{and}\quad b_j(\alpha_i)\in\textstyle\bigwedge^{i-j}F^*. \tag{1.1}$$

(We view $\bigwedge^i F$ and D_iF to be meaningful for every integer i; in particular, these modules are zero whenever i is negative.) The exterior and divided power algebras A come equipped with co-multiplication $\Delta\colon A\to A\otimes A$. The following facts are well known; see [7, section 1], [8, Appendix], and [14, section 1].

Proposition 1.2. *Let F be a free module of rank f over a commutative noetherian ring R and let $b_r\in\bigwedge^r F$, $b'_p\in\bigwedge^p F$, and $\alpha_q\in\bigwedge^q F^*$.*

(a) *If $p=f$, then $[b_r(\alpha_q)](b'_p)=b_r\wedge\alpha_q(b'_p)$.*

(b) *If $X\colon F\to G$ is a homomorphism of free R-modules and $\gamma_{s+r}\in\bigwedge^{s+r}G^*$, then* $(\bigwedge^s X^*)\left[\left((\bigwedge^r X)(b_r)\right)(\gamma_{s+r})\right]=b_r\left[\left(\bigwedge^{s+r}X^*\right)(\gamma_{s+r})\right]$.

Notation. Let m be an integer. Each pair of elements (U,Y), with $U\in D_mE$ and $Y\in\bigwedge^m G^*$, gives rise to an element of $\bigwedge^m(E\otimes G^*)$, which we denote by $U\bowtie Y$. We now give the definition of $U\bowtie Y$. Consider the composition

$$D_mE\otimes T_mG^*\xrightarrow{\Delta\otimes 1}T_mE\otimes T_mG^*\xrightarrow{\psi}\bigwedge^m(E\otimes G^*),$$

where $\psi\Big((U_1\otimes\cdots\otimes U_m)\otimes(Y_1\otimes\cdots\otimes Y_m)\Big)=(U_1\otimes Y_1)\wedge\cdots\wedge(U_m\otimes Y_m)$, for $U_i\in E$ and $Y_i\in G^*$. It is easy to see that the above composition factors through $D_mE\otimes\bigwedge^m G^*$. Let $U\otimes Y\mapsto U\bowtie Y$ be the resulting map from $D_mE\otimes\bigwedge^m G^*$ to $\bigwedge^m(E\otimes G^*)$. Notice, for example, that if $u\in E$ and $Y_i\in G^*$, then

$$u^{(m)}\bowtie(Y_1\wedge\cdots\wedge Y_m)=(u\otimes Y_1)\wedge\cdots\wedge(u\otimes Y_m).$$

The map

$$\bigwedge^m E\otimes D_mG^*\to\bigwedge^m(E\otimes G^*),$$

which sends $U\otimes Y$ to $U\bowtie Y$, for $U\in\bigwedge^m E$ and $Y\in D_mG^*$, is defined in a completely analogous manner.

Definition. If $Y\colon E\to G$ is a homomorphism of free R-modules of finite rank, then let $\check{Y}$ be the element of $(E\otimes G^*)^*$ which corresponds to Y under the adjoint isomorphism. In other words, $\check{Y}(\varepsilon\otimes\gamma)=[Y(\varepsilon)](\gamma)$. In light of (1.1), we view $\check{Y}$ as a differential on the exterior algebra $\bigwedge^\bullet(E\otimes G^*)$.

Remark. If one thinks of Y as a matrix and takes ε and γ to be basis elements of E and G^*, respectively, then $\check{Y}(\varepsilon\otimes\gamma)$ is the corresponding entry of Y. The differential graded algebra $\big(\bigwedge^\bullet(E\otimes G^*),\check{Y}\big)$ is the "Koszul complex" associated to the entries of a matrix representation of Y.

Lemma 1.3. *Suppose R is a polynomial ring over the ring of integers, E and M are free $R-$modules, and $\varphi\colon D_rE\to M$ is an $R-$module homomorphism. If $\varphi(\varepsilon_1^{(r)})=0$ for all $\varepsilon_1\in E$, then φ is identically zero.*

Remarks. If $R\to\bar{R}$ is any base change, then $\varphi\otimes 1_{\bar{R}}$ is also identically zero. On the other hand, the lemma would be false if R were allowed to have torsion. Indeed, if $R=\mathbb{Z}/(2)$, $E=Rx\oplus Ry$ has rank 2, and M has rank 1, then $\varphi\colon D_3E\to R$, given by $\varphi(x^{(3)})=\varphi(y^{(3)})=0$ and $\varphi(xy^{(2)})=\varphi(x^{(2)}y)=1$, defines an R-module homomorphism with $\varphi(\varepsilon_1^{(3)})=0$ for all $\varepsilon_1\in E$, but φ is not identically zero.

Proof. Every element of D_rE is a linear combination of elements of the form

$$\varepsilon_1^{(a_1)}\cdots\varepsilon_s^{(a_s)}$$

for some positive integers s, and $a_1, \dots, a_s$, with $a_1 + \cdots + a_s = r$, and elements $\varepsilon_1, \dots, \varepsilon_s$ in E. We show that $D_r(E) \subseteq \ker\varphi$ by induction on s. The case $s = 1$ is the original hypothesis. Suppose that all elements of the above form are in $\ker\varphi$ for some s. Fix the element $Y = \varepsilon_1^{(a_1)} \cdots \varepsilon_s^{(a_s)} \varepsilon_{s+1}^{(a_{s+1})}$ of $D_r E$. Let $a = a_s + a_{s+1}$, and X be the element $\varepsilon_1^{(a_1)} \cdots \varepsilon_{s-1}^{(a_{s-1})}$ of $D_{r-a}E$. The induction hypothesis ensures that for each integer n, $X(\varepsilon_s + n\varepsilon_{s+1})^{(a)}$ is in $\ker\varphi$. We see that $X(\varepsilon_s + n\varepsilon_{s+1})^{(a)}$ is equal to the product

$$[\,1 \quad n \quad n^2 \quad \dots \quad n^a\,] \begin{bmatrix} X\varepsilon_s^{(a)}\varepsilon_{s+1}^{(0)} \\ X\varepsilon_s^{(a-1)}\varepsilon_{s+1}^{(1)} \\ X\varepsilon_s^{(a-2)}\varepsilon_{s+1}^{(2)} \\ \vdots \\ X\varepsilon_s^{(0)}\varepsilon_{s+1}^{(a)} \end{bmatrix}.$$

The row vector in the above product is a row from a Vandermonde matrix. A matrix argument produces a non-zero integer N so that $NX\varepsilon_s^{(a-i)}\varepsilon_{s+1}^{(i)} \in \ker\varphi$, for all i, with $0 \le i \le a$. It follows that $N\varphi(Y) = 0$ in the free abelian group M; so, $\varphi(Y) = 0$. $\square$

Convention. If F is a free module of rank f, then we orient F by fixing basis elements $\omega_F \in \bigwedge^f F$ and $\omega_{F^*} \in \bigwedge^f F^*$, which are compatible in the sense that $\omega_F(\omega_{F^*}) = 1$.

Convention. For each statement "S", we define

$$\chi(\mathrm{S}) = \begin{cases} 1, & \text{if S is true, and} \\ 0, & \text{if S is false.} \end{cases}$$

In particular, $\chi(i = j)$ has the same value as the Kronecker delta δ_{ij}.

2. The complex $\mathbb{F}$.

Data 2.1. Let R be a commutative noetherian ring and let e, f, and g be positive integers which satisfy $f = e + g$. The complex $\mathbb{F}$ is built from data $(\mathfrak{b}, V, X)$ where $\mathfrak{b}$ is an element of R, and V and X are R-module homomorphisms:

$$E \xrightarrow{V} F \xrightarrow{X} G,$$

with E, F and G free R-modules of rank e, f, and g, respectively. For integers a, c, and d, define

$$A(a, c, d) = D_a E \otimes D_c G^* \otimes \bigwedge^d (E \otimes G^*) \otimes \bigwedge^{a-c+e} F^* \text{ and } B(d) = \bigwedge^d (E \otimes G^*).$$

Remark. If $V = [v_{jk}]$ and $X = [x_{ij}]$ are matrices, with $1 \le k \le e$, $1 \le j \le f$, and $1 \le i \le g$, and R is the polynomial ring $R_0[\{\mathfrak{b}\} \cup \{v_{jk}\} \cup \{x_{ij}\}]$, where $\{\mathfrak{b}\} \cup \{v_{jk}\} \cup \{x_{ij}\}$ is a list of indeterminates over a commutative noetherian ring R_0, then we say that the data of 2.1 is generic.

Grading Convention 2.2. Let $\mathcal{A}$ be the additive sub-monoid of $\mathbb{Z}^2$ which is generated by $(1,0)$, $(0,1)$, and $(-e,g)$. Notice that

$$0 \text{ is the only invertible element of } \mathcal{A}. \tag{2.3}$$

If the data of 2.1 is generic, then the polynomial ring $R = \bigoplus_{a\in\mathcal{A}} R_a$ is graded by $\mathcal{A}$, where the variables v_{jk}, x_{ij}, and $\mathfrak{b}$ of R are elements of $R_{(1,0)}$, $R_{(0,1)}$, and $R_{(-e,g)}$, respectively. Furthermore, if $I(R)$ is the R-submodule $\bigoplus_{a\neq 0} R_a$ of R, then condition (2.3) ensures that $I(R)$ is an ideal of R and $R/I(R) \cong R_0$.

Remark. A different approach to the grading of R is given by choosing positive integers d_x and d_v with $gd_x - ed_v$ also positive; for example Tchernev, takes $(\deg \mathfrak{b}, d_v, d_x) = (g, g, e+1)$. Then the degrees of v_{jk}, x_{ij}, and $\mathfrak{b}$ are d_v, d_x, and $gd_x - ed_v$, respectively.

Definition 2.4. Let $(\mathfrak{b}, V, X)$ be the data of 2.1. For each integer i, the module $\mathbb{F}_i$ in the complex $(\mathbb{F}, \boldsymbol{d})$ is

$$\mathbb{F}_i = B(i) \oplus \bigoplus_{(a,c,d)} A(a,c,d),$$

where the parameters satisfy $i = a + c + d + 1$. The differential $\boldsymbol{d}\colon \mathbb{F}_i \to \mathbb{F}_{i-1}$ is defined as follows. If $z_d \in B(d)$, then

$$\boldsymbol{d}(z_d) = (X \check{\circ} V)(z_d) \in B(d-1), \text{ and}$$

if $x = \varepsilon_1^{(a)} \otimes \gamma_1^{(c)} \otimes z_d \otimes \alpha_b \in A(a,c,d)$, with $b = a - c + e$, then $\boldsymbol{d}(x)$ is equal to

$$\begin{cases} \varepsilon_1^{(a-1)} \otimes \gamma_1^{(c)} \otimes z_d \otimes [V(\varepsilon_1)](\alpha_b) \in A(a-1,c,d) \\ -\varepsilon_1^{(a)} \otimes \gamma_1^{(c-1)} \otimes z_d \otimes X^*(\gamma_1) \wedge \alpha_b \in A(a,c-1,d) \\ +(-1)^{a+c}\varepsilon_1^{(a)} \otimes \gamma_1^{(c)} \otimes (X \check{\circ} V)(z_d) \otimes \alpha_b \in A(a,c,d-1) \\ +(-1)^{a+c}\varepsilon_1^{(a-1)} \otimes \gamma_1^{(c-1)} \otimes (\varepsilon_1 \otimes \gamma_1) \wedge z_d \otimes \alpha_b \in A(a-1,c-1,d+1) \\ +(-1)^{a+d}\chi(c=0)\varepsilon_1^{(a)} \bowtie \left[(\bigwedge^{f-b} X)(\alpha_b[\omega_F])\right](\omega_{G^*}) \wedge z_d \in B(a+d) \\ +(-1)^d\chi(a=0)\mathfrak{b} \cdot [(\bigwedge^b V^*)(\alpha_b)](\omega_E) \bowtie \gamma_1^{(c)} \wedge z_d \in B(c+d). \end{cases}$$

The map

$$A(a,c,d) \to A(a-1,c,d)$$

is the composition

$$\begin{aligned} D_aE\otimes M\otimes {\textstyle\bigwedge}^b F^* &\xrightarrow{\Delta\otimes 1\otimes 1} D_{a-1}E\otimes E\otimes M\otimes {\textstyle\bigwedge}^b F^* \xrightarrow{1\otimes V\otimes 1\otimes 1} D_{a-1}E\otimes F\otimes M\otimes {\textstyle\bigwedge}^b F^* \\ &\longrightarrow D_{a-1}E\otimes M\otimes F\otimes {\textstyle\bigwedge}^b F^* \xrightarrow{1\otimes(1.1)\otimes 1} D_{a-1}E\otimes M\otimes {\textstyle\bigwedge}^{b-1}F^*, \end{aligned}$$

where $M = D_cG^*\otimes \bigwedge^d(E\otimes G^*)$. The maps from $A(a,c,d)$ to $A(a,c-1,d)$ and $A(a-1,c-1,d+1)$ also involve co-multiplication in a divided power algebra.

Remark 2.5. If the data of 2.1 satisfies the grading convention of 2.2, then the complex $\mathbb{F}$ is homogeneous in the $\mathcal{A}$-grading, provided

$$\mathbb{F}_i = B(i)[-i,-i]\oplus \bigoplus_{(a,c,d)} A(a,c,d)[-a-d,-g-c-d],$$

where the sum is taken over all parameters (a,c,d) which satisfy $i=a+c+d+1$.

Proposition. *The maps and modules $(\mathbb{F},\boldsymbol{d})$ of Definition 2.4 form a complex*

Proof. In light of Lemma 1.3 it suffices to show that $\boldsymbol{d}\circ\boldsymbol{d}(x)=0$ for

$$x=\varepsilon_1^{(a)}\otimes\gamma_1^{(c)}\otimes z_d\otimes\alpha_b\in A(a,c,d)$$

and $b=a-c+e$. The calculation is routine. We pick out a couple of high points. The observation that if $Y\colon E\to G$, then

$$\check{Y}(\varepsilon_k\bowtie\gamma_1^{(k)}) = \Big([Y^*(\gamma_1)](\varepsilon_k)\Big)\bowtie\gamma_1^{(k-1)}$$

for all $\varepsilon_k\in\bigwedge^k E$ and $\gamma_1\in G^*$, is the key to seeing that that the $B(a+d-1)$ component of $\boldsymbol{d}\circ\boldsymbol{d}(x)$ is zero, when $c=0$, and that the $B(c+d-1)$ component of $\boldsymbol{d}\circ\boldsymbol{d}(x)$ is zero, when $a=0$. In the $B(c+d)$ component of $\boldsymbol{d}\circ\boldsymbol{d}(x)$, when $a=1$, we use Proposition 1.2 (b) and (a) to see that

$$\begin{aligned} &[({\textstyle\bigwedge}^{b-1}V^*)([V(\varepsilon_1)](\alpha_b))](\omega_E)\bowtie\gamma_1^{(c)} = \Big(\varepsilon_1[({\textstyle\bigwedge}^b V^*)(\alpha_b)]\Big)(\omega_E)\bowtie\gamma_1^{(c)} \\ &= \Big(\varepsilon_1\wedge[({\textstyle\bigwedge}^b V^*)(\alpha_b)](\omega_E)\Big)\bowtie\gamma_1^{(c)} = (\varepsilon_1\otimes\gamma_1)\wedge\Big([({\textstyle\bigwedge}^b V^*)(\alpha_b)](\omega_E)\bowtie\gamma_1^{(c-1)}\Big). \end{aligned}$$

The same type of argument gives

$$\Big[({\textstyle\bigwedge}^{f-b-1}X)([X^*(\gamma_1)\wedge\alpha_b][\omega_F])\Big](\omega_{G^*}) = \gamma_1\wedge\Big(({\textstyle\bigwedge}^{f-b}X)(\alpha_b[\omega_F])\Big)(\omega_{G^*}),$$

which is the key to seeing that the $B(a+d)$ component of $\boldsymbol{d}\circ\boldsymbol{d}(x)$ is equal to zero when $c=1$. $\square$

Definition 2.6. Let $(\mathfrak{b}, V, X)$ be the data of 2.1. Define λ to be the element

$$(-1)^{eg}[(\bigwedge^g X^*)(\omega_{G^*})](\omega_F) + \mathfrak{b}(\bigwedge^e V)(\omega_E)$$

of $\bigwedge^e F$ and define J to be the image of the map

$$[\,\lambda \quad (X \check{\circ} V)\,] : \bigwedge^e F^* \oplus (E \otimes G^*) \to R.$$

Observation. *If $(\mathbb{F}, \boldsymbol{d})$ is the complex of 2.4 and J is the ideal of Definition 2.6, then the homology $\mathrm{H}_0(\mathbb{F})$ is equal to R/J. Furthermore, if $(\mathbb{F}, \boldsymbol{d})$ is formed using the polynomial ring $\mathcal{P}$ and the data $(\mathfrak{b}, V, X)$ of (0.2), then the homology $\mathrm{H}_0(\mathbb{F})$ is equal to the universal ring $\mathcal{R} = \mathcal{P}/\mathcal{J}$.*

Proof. The beginning of $\mathbb{F}$ is $\mathbb{F}_1 \to \mathbb{F}_0 \to 0$, with

$$\mathbb{F}_1 = A(0,0,0) \oplus B(1) = \bigwedge^e F^* \oplus (E \otimes G^*) \quad \text{and} \quad \mathbb{F}_0 = B(0) = R.$$

The map $E \otimes G^* \to R$ is $(X \check{\circ} V)$, and the element $\alpha_e \in \bigwedge^e F^*$ is sent to

$$[(\bigwedge^g X)(\alpha_e(\omega_F)]\,(\omega_{G^*}) + \mathfrak{b}\,[(\bigwedge^e V^*)(\alpha_e)]\,(\omega_E) = \lambda(\alpha_e) \in R.$$

The first assertion is established. The homomorphisms X and V, of the second assertion, are represented by matrices. Let α_e be a basis vector in $\bigwedge^e F^*$. The element $[(\bigwedge^e V^*)(\alpha_e)]\,(\omega_E)$ of R is the determinant of the submatrix of V determined by the e rows picked out by α_e. The element $\alpha_e(\omega_F)$ of $\bigwedge^g F$ picks out the complementary columns of X with the correct sign, and $[(\bigwedge^g X)(\alpha_e(\omega_F)]\,(\omega_{G^*})$ is the (signed) determinant of this submatrix of X. $\square$

In Theorem 6.1 we prove that $\mathbb{F}$ is acyclic whenever the data is generic. However, $\mathbb{F}$ is far from a minimal resolution. On the other hand, it is possible to isolate the part of $\mathbb{F}$ in which the splitting occurs. To do this, we partition $\mathbb{F}$ into strands. Our definition of the strands is motivated by Remark 2.5.

Definition 2.7. Let $(\mathbb{F}, \boldsymbol{d})$ be the complex of Definition 2.4 and let P and Q be integers. The module $A(a,c,d)$ from $\mathbb{F}$ is in the strand $S(P,Q)$ if $P = a + d$ and $Q = c + d$. The module $B(d)$ is in $S(P,Q)$ if $P = d$ and $Q = d - g$. We impose the *inverse lexicographic order* on $\mathbb{Z} \times \mathbb{Z}$. In other words, $(P', Q') \le (P, Q)$, if $Q' < Q$, or if $Q' = Q$ and $P' \le P$.

Observation 2.8.
(a) *If the strand* $S(P,Q)$ *is non-zero, then* $0 \le P$ *and* $-g \le Q-P \le e$.
(b) *As a module,* $\mathbb{F} = \bigoplus_{(P,Q)} S(P,Q)$.
(c) *The decomposition of* (b) *satisfies hypothesis 4.1.*

Proof. The first assertion holds because if $A(a,c,d)$ is a non-zero summand of $S(P,Q)$, then $0 \le b \le f$ and $b = a-c+e = P-Q+e$. Assertion (b) is obvious. If $P = d$ and $Q = d-g$, then $B(d) \subset S(P,Q)$ and $\boldsymbol{d}(B(d)) \subset S(P-1,Q-1)$. If $a+d = P$ and $c+d = Q$, then $A(a,c,d) \subset S(P,Q)$ and

$$\boldsymbol{d}(A(a,c,d)) \subset \begin{cases} S(P-1,Q) \oplus S(P,Q-1) \oplus S(P-1,Q-1) \oplus S(P,Q) \\ \oplus\, \chi(c=0) S(P,Q+a-g) \oplus \chi(a=0) S(P+c,Q-g). \end{cases}$$

We have already seen that if $c = 0$, then $a - g \le 0$. Assertion (c) has been established. $\square$

For each (P,Q) in the poset $\mathbb{Z} \times \mathbb{Z}$, let $(S(P,Q), \boldsymbol{\partial})$ be the homogeneous strand of $(\mathbb{F}, \boldsymbol{d})$ which is induced by the direct sum decomposition of (b) as described in 4.1. Let $(\mathbb{F}, \boldsymbol{\partial})$ be the direct sum of the all of the strands $(S(P,Q), \boldsymbol{\partial})$.

Example 2.9. Fix integers P and Q. If $P - g < Q$, then $S(P,Q)$ is

$$0 \to A(P,Q,0) \xrightarrow{\boldsymbol{\partial}} A(P-1,Q-1,1) \xrightarrow{\boldsymbol{\partial}} \dots \xrightarrow{\boldsymbol{\partial}} A(P-eg, Q-eg, eg) \to 0.$$

If $Q = P - g$, then $S(P,Q)$ is

$$0 \to A(P,Q,0) \xrightarrow{\boldsymbol{\partial}} A(P-1,Q-1,1) \xrightarrow{\boldsymbol{\partial}} \dots \xrightarrow{\boldsymbol{\partial}} A(P-Q,0,Q) \xrightarrow{\boldsymbol{\partial}} B(P) \to 0.$$

In all cases the module $A(a,c,d)$ is in position $a+c+d+1$.

3. The case $e = 1$.

Theorem 3.1. *The complex* $(\mathbb{F}, \boldsymbol{d})$ *of Definition 2.4 is acyclic when the data is generic and* $e = 1$.

Proof. In Definition 3.3 and Proposition 3.4, we produce $q\colon \mathbb{P} \to \mathbb{P}'$, which is a map of acyclic complexes. We define a map of complexes $\varphi\colon \mathbb{F} \to \mathbb{P}$ in Proposition 3.5. It is clear that $q \circ \varphi\colon \mathbb{F} \to \mathbb{P}'$ is surjective. In Proposition 3.7 we identify the kernel of $q \circ \varphi$ as $M + \boldsymbol{d}M$. Lemma 3.9 gives an isomorphism of complexes $\Theta\colon (\mathbb{F}, \boldsymbol{d}) \to (\mathbb{F}, \boldsymbol{D})$ which carries $M + \boldsymbol{d}(M)$ to $M + \boldsymbol{D}(M)$. This lemma also shows that $M + \boldsymbol{D}(M)$ is split exact. It follows that $M + \boldsymbol{d}(M) = \ker(q \circ \varphi)$ is split exact; and therefore, $\mathrm{H}_i(\mathbb{F}) = \mathrm{H}_i(\mathbb{P}')$ for all i; thus, $\mathbb{F}$ is acyclic. $\square$

Data 3.2. Let R be a commutative noetherian ring, g be a positive integer, and $f = g+1$. The complex $\mathbb{P}$ is built from data $(\mathfrak{b}, v, X)$, where $\mathfrak{b}$ is an element of R, $X\colon F \to G$ is an R-module homomorphism, with F and G free R-modules of rank f and g, respectively, and v is an element of F.

Remark. If we think as the data of 3.2 as matrices $v = [v_{j1}]$ and $X = [x_{ij}]$, with $1 \le j \le f$ and $1 \le i \le g$, and R is the polynomial ring $R_0[\{\mathfrak{b}\} \cup \{v_{j1}\} \cup \{x_{ij}\}]$, where $\{\mathfrak{b}\} \cup \{v_{j1}\} \cup \{x_{ij}\}$ is a list of indeterminates over a ring R_0, then we say that the data of 3.2 is generic.

Definition 3.3. Adopt the data $(\mathfrak{b}, v, X)$ of 3.2. The complex $(\mathbb{P}, \boldsymbol{d})$, has modules $\mathbb{P}_i = \bigwedge^i G^* \oplus \bigwedge^i F^* \oplus \bigwedge^{i-1} G^*$ and differential

$$\boldsymbol{d}_i = \begin{bmatrix} X(v) & B_i & (-1)^{i+1}\mathfrak{b} \\ 0 & -v & A_{i-1} \\ 0 & 0 & X(v) \end{bmatrix},$$

where the maps $A_i\colon \bigwedge^i G^* \to \bigwedge^i F^*$ and $B_i\colon \bigwedge^i F^* \to \bigwedge^{i-1} G^*$ are given by $A_i(\gamma_i) = \bigwedge^i X^*(\gamma_i)$ and $B_i(\alpha_i) = [(\bigwedge^{f-i} X)(\alpha_i[\omega_F])](\omega_{G^*})$. The complex $(\mathbb{P}', \boldsymbol{d}')$ is the same as the complex $(\mathbb{P}, \boldsymbol{d})$, except

$$\mathbb{P}'_1 = \textstyle\bigwedge^1 G^* \oplus \bigwedge^1 F^*, \quad \mathbb{P}'_0 = R, \quad \boldsymbol{d}'_2 = \begin{bmatrix} X(v) & B_2 & -\mathfrak{b} \\ 0 & -v & A_1 \end{bmatrix},$$

and $\boldsymbol{d}'_1 = [\, X(v) \quad B_1 + \mathfrak{b}v \,]$. The map of complexes $q\colon \mathbb{P} \to \mathbb{P}'$ be given by q_i is the identity map for $2 \le i$,

$$q_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \text{and} \quad q_0 = [\, 1 \quad -\mathfrak{b} \,].$$

Proposition 3.4. *If the data $(\mathfrak{b}, v, X)$ of 3.2 is generic, then the complexes $\mathbb{P}$ and $\mathbb{P}'$ are acyclic.*

Proof. One may prove this directly or see [18, Prop. 1.2] or [17, Theorem 1.3]. □

Proposition 3.5. *Let $(\mathbb{F}, \boldsymbol{d})$ be the complex of Definition 2.4 constructed from data $(\mathfrak{b}, V, X)$ with $e = 1$, and let $\mathbb{P}$ be the complex of Definition 3.3 constructed from $(\mathfrak{b}, v, X)$, where $v = V(1) \in F$. If the function $\varphi\colon \mathbb{F} \to \mathbb{P}$ is defined by*

$$\varphi(\gamma_d) = \begin{bmatrix} \gamma_d \\ 0 \\ 0 \end{bmatrix} \quad \textit{and} \quad \varphi(x) = \begin{bmatrix} 0 \\ \chi(0 \le a)\chi(0 = c)(-1)^a \alpha_b \wedge (\bigwedge^d X^*)(\gamma_d) \\ \chi(0 = a)(-1)^c (\bigwedge^b v)(\alpha_b) \cdot \gamma_1^{(c)} \wedge \gamma_d \end{bmatrix},$$

for $\gamma_d \in B(d)$ *and* $x = 1^{(a)} \otimes \gamma_1^{(c)} \otimes \gamma_d \otimes \alpha_b \in A(a,c,d)$*, then* φ *is a map of complexes.*

Proof. We pick out one high point of this calculation. Fix a, b, c, and d with $i = a+c+d+1$ and $b = a-c+1$. Let

$$y = (\boldsymbol{d}_i \circ \varphi_i - \varphi_{i-1} \circ \boldsymbol{d}_i)(x) \in \mathbb{P}_{i-1} = \bigwedge^{i-1} G^* \oplus \bigwedge^{i-1} F^* \oplus \bigwedge^{i-2} G^*.$$

When $c = 0$, the $\bigwedge^{i-1} G^*$ component of y is zero because

$$\begin{aligned} B_i\left(\alpha_b \wedge (\textstyle\bigwedge^d X^*)(\gamma_d)\right) &= \left((\textstyle\bigwedge^{f-i} X)\left[\left(\alpha_b \wedge (\textstyle\bigwedge^d X^*)(\gamma_d)\right)(\omega_F)\right]\right)(\omega_{G^*}) \\ &= (-1)^{bd}\left((\textstyle\bigwedge^{f-i} X)\left[\left((\textstyle\bigwedge^d X^*)(\gamma_d)\right)(\alpha_b(\omega_F))\right]\right)(\omega_{G^*}) \\ &= (-1)^{bd}\gamma_d \wedge \left[(\textstyle\bigwedge^{f-b} X)(\alpha_b(\omega_F))\right](\omega_{G^*}). \end{aligned}$$

The final equality follows from Proposition 1.2 (b) and (a). $\square$

Lemma 3.6. *Let* $(\mathbb{F}, \boldsymbol{d})$ *be the complex of Definition 2.4 constructed from data* $(\mathfrak{b}, V, X)$ *with* $e = 1$*. If* $1 \le i$*, then there are submodules* $K(0,1,i-1)$ *and* $L(0,1,i-1)$ *of* $A(0,1,i-1)$ *and a homomorphism* $s_i \colon \bigwedge^i G^* \to A(0,1,i-1)$ *such that*

(a) $A(0,1,i-1) = K(0,1,i-1) \oplus L(0,1,i-1)$,
(b) $K(0,1,i-1)$ *is the image of* $\boldsymbol{\partial} \colon A(1,2,i-2) \to A(0,1,i-1)$ (*see Example 2.9*),
(c) *exterior multiplication carries* $L(0,1,i-1)$ *isomorphically onto* $\bigwedge^i G^*$*, and*
(d) s_i *is a splitting of the exterior multiplication map* $\mu \colon A(0,1,i-1) \to \bigwedge^i G^*$.

Proof. The module $A(0,1,i-1)$ is equal to $G^* \otimes \bigwedge^{i-1} G^*$. The map μ of (d) is surjective; hence, there exists $s_i \colon \bigwedge^i G^* \to A(0,1,i-1)$ with $\mu \circ s_i$ equal to the identity map on $\bigwedge^i G^*$. Let $L(0,1,i-1) = \operatorname{im} s_i$ and $K(0,1,i-1) = \ker \mu$. The decomposition $A(0,1,i-1) = \ker \mu \oplus \operatorname{im} s_i$ gives (a). To complete the proof, recall that the complex

$$\cdots \to D_2 G^* \otimes \bigwedge^{i-2} G^* \to D_1 G^* \otimes \bigwedge^{i-1} G^* \to D_0 G^* \otimes \bigwedge^i G^* \to 0$$

is split exact for $1 \le i$. $\square$

Proposition 3.7. *Adopt the notation and hypotheses of Proposition 3.5. Let* M *be the submodule* $M = \bigoplus A(a,c,d)$ *of* $\mathbb{F}$*, where the sum is taken over all* 3*-tuples* (a,c,d)*, with* $1 \le a$ *and* $1 \le c$*. Then* $\ker(q \circ \varphi) = M + \boldsymbol{d}(M)$.

Proof. It is clear that $M+\boldsymbol{d}(M)\subseteq \ker(q\circ\varphi)$. It is not difficult to see that $\ker(q\circ\varphi)_i=0$ for $0\le i\le 1$, and $M_i=0$ for $2\le i$. Henceforth, we take $2\le i$. We next show that

$$\mathbb{F}_i = M_i+\boldsymbol{d}(M_{i+1})+B(i)+A(i-1,0,0)+L(0,1,i-2), \tag{3.8}$$

where $L(0,1,i-2)$ is defined in Lemma 3.6. First of all, it is easy to see that

$$\mathbb{F}_i = M_i+B(i)+A(0,0,i-1)+A(0,1,i-2)+\sum_{a=1}^{i-1}A(a,0,i-1-a).$$

Indeed, if $A(a,c,d)$ is a summand of $\mathbb{F}_i$, but is not a summand of M_i, then either $a=0$ or $c=0$. If $a=0$, then either $c=0$ (in which case $(a,c,d)=(0,0,i-1)$) or $c=1$ (in which case $(a,c,d)=(0,1,i-2)$). If $c=0$ and $1\le a$, then $A(a,c,d)$ is a summand of $\sum\limits_{a=1}^{i-1}A(a,0,i-1-a)$. Apply $\boldsymbol{d}$ to $A(1,1,i-2)$, which is a summand of M_{i+1}, to see that

$$A(0,0,i-1)\subseteq \boldsymbol{d}(M_{i+1})+M_i+A(0,1,i-2)+A(1,0,i-2).$$

Apply $\boldsymbol{d}$ to $A(1,2,i-3)$, which is a summand of M_{i+1}, to see that

$$A(0,1,i-2)\subseteq L(0,1,i-2)+\boldsymbol{d}(M_{i+1})+M_i.$$

If $1\le a\le i-2$, then apply $\boldsymbol{d}$ to $A(a+1,1,i-2-a)$, which is a summand of M_{i+1}, to see that

$$A(a,0,i-1-a)\subseteq M_i+\boldsymbol{d}(M_{i+1})+A(a+1,0,i-2-a).$$

Let

$$P_i=\frac{B(i)\oplus A(i-1,0,0)\oplus L(0,1,i-2)}{[B(i)\oplus A(i-1,0,0)\oplus L(0,1,i-2)]\cap[M_i+\boldsymbol{d}(M_{i+1})]}.$$

Now that (3.8) is established, we know that $\mathbb{F}_i/[M_i+\boldsymbol{d}(M_{i+1})]\cong P_i$. On the other hand, the composition

$$\mathbb{P}_i\cong B(i)\oplus A(i-1,0,0)\oplus L(0,1,i-2)\xrightarrow{\text{nat}} P_i\cong \frac{\mathbb{F}_i}{M_i+\boldsymbol{d}(M_{i+1})}\xrightarrow{\varphi_i}\mathbb{P}_i$$

is an isomorphism, where nat is the natural quotient map. It follows that

$$[B(i)\oplus A(i-1,0,0)\oplus L(0,1,i-2)]\cap[M_i+\boldsymbol{d}(M_{i+1})]=0,$$

and $\ker(q\circ\varphi)=M+\boldsymbol{d}(M)$. $\square$

Lemma 3.9. *Adopt the notation and hypotheses of Proposition 3.7. There exists a differential $\boldsymbol{D}$ on $\mathbb{F}$ and a module automorphism Θ of $\mathbb{F}$, such that*
(a) $(\mathbb{F},\boldsymbol{D})$ *is a complex,*
(b) $\Theta\colon(\mathbb{F},\boldsymbol{d})\to(\mathbb{F},\boldsymbol{D})$ *is an isomorphism of complexes,*
(c) Θ *acts like the identity map on* M*,*
(d) $\Theta(M+\boldsymbol{d}(M))=M+\boldsymbol{D}(M)$*, and*
(e) $M+\boldsymbol{D}(M)$ *is split exact.*

Proof. Recall s_d and $K(0,1,d)$ from Lemma 3.6. We define $\Theta\colon\mathbb{F}\to\mathbb{F}$. The map Θ acts like the identity on each $B(i)$. If

$$x=1^{(a)}\otimes\gamma_1^{(c)}\otimes\gamma_d\otimes\alpha_b\in A(a,c,d),$$

then $\Theta(x)$ is equal to

$$\begin{cases}+1^{(a)}\otimes\gamma_1^{(c)}\otimes\gamma_d\otimes\alpha_b\in A(a,c,d)\\ +\chi(c=0)\chi(1\le d)(-1)^d1^{(a+d)}\otimes\gamma_1^{(c)}\otimes 1\otimes\alpha_b\wedge\bigwedge^d X^*(\gamma_d)\in A(a+d,c,0)\\ -\chi(a=0)\chi(c=0)1^{(a)}\otimes s_d(\gamma_d)\otimes v(\alpha_b)\in A(a,c+1,d-1).\end{cases}$$

Let $\boldsymbol{D}_i=\Theta_{i-1}\circ\boldsymbol{d}_i\circ\Theta_i^{-1}$. Assertions (a), (b), and (c) are established. Assertion (d) follows from (b) and (c). If the element x, from the above display, is in M_i, then a straightforward calculation shows that $\boldsymbol{D}_i(x)$ is equal to

$$\begin{cases}+\chi(2\le a)1^{(a-1)}\otimes\gamma_1^{(c)}\otimes\gamma_d\otimes v(\alpha_b)\in A(a-1,c,d)\\ +\chi(a=1)\chi(c=1)1^{(a-1)}\otimes(\mathrm{id}-s\circ\mu)(\gamma_1\otimes\gamma_d)\otimes v(\alpha_b)\in K(0,1,d)\\ -[\chi(2\le c)+\chi(c=1)\chi(1\le d)]1^{(a)}\otimes\gamma_1^{(c-1)}\otimes\gamma_d\otimes X^*(\gamma_1)\wedge\alpha_b\in A(a,c-1,d)\\ +(-1)^{a+c}1^{(a)}\otimes\gamma_1^{(c)}\otimes[X(v)](\gamma_d)\otimes\alpha_b\in A(a,c,d-1)\\ +(-1)^{a+c}1^{(a-1)}\otimes\gamma_1^{(c-1)}\otimes\gamma_1\wedge\gamma_d\otimes\alpha_b\in A(a-1,c-1,d+1).\end{cases}$$

For each i, with $2\le i$, let

$$\mathbb{S}_i=M_i\oplus K(0,1,i-2)\oplus\sum_{(a,d)}A(a,0,d),$$

where the sum is taken over all pairs (a,d) with $0\le a$, $1\le d$, and $a+d+1=i$. Observe that $\boldsymbol{D}(M_i)\subseteq\mathbb{S}_{i-1}$. The only term which causes any effort is the term in $A=A(a-1,c-1,d+1)$. If $a-1=0$, then $b=a-c+e$ forces $c\le 2$. On the other hand, A is zero unless $1\le c$. If $c=1$, then $A=A(0,0,d+1)\subseteq\mathbb{S}_{i-1}$. If $c=2$, then $A=A(0,1,d+1)$, but it is clear that

$$(-1)^{a+c}1^{(a-1)}\otimes\gamma_1^{(c-1)}\otimes\gamma_1\wedge\gamma_d\otimes\alpha_b\in K(0,1,d+1)\subseteq\mathbb{S}_{i-1}.$$

If $1 \le a-1$, then either A is in $M_{i-1} \subseteq \mathbb{S}_{i-1}$ or $c=1$, in which case, we still have $A = A(a-1,0,d+1) \subseteq \mathbb{S}_{i-1}$.

Let $\mathbb{S} = \bigoplus_{2\le i} \mathbb{S}_i$. We have shown that $\boldsymbol{D}(M_i) \subseteq \mathbb{S}_{i-1}$; hence $M + \boldsymbol{D}(M) \subseteq \mathbb{S}$. We complete the proof by showing that $(\mathbb{S}, \boldsymbol{D})$ is split exact and $\mathbb{S} \subseteq M + \boldsymbol{D}(M)$. In a manner analogous to Definition 2.7, we partition $\mathbb{S}$ into strands $\bigoplus_{(P,Q)} \bar{S}(P,Q)$, where the sum varies over all pairs (P,Q) with $1 \le P$ and $1 \le Q \le P+1$. For parameters P and Q, the summand $X(a,c,d)$ of $\mathbb{S}$ is in $\bar{S}(P,Q)$ if $P = a+d$ and $Q = c+d$, where $X = A$ for all (a,c,d), except $(0,1,d)$, and $X = K$ for $(a,c,d) = (0,1,d)$. Observe that every summand of $\mathbb{S}$ lives in exactly one strand. Our calculation of $\boldsymbol{D}(x)$, for $x \in M_i$, shows that the decomposition $\mathbb{S} = \bigoplus \bar{S}(P,Q)$ satisfies the hypothesis of 4.1. Furthermore, the homogeneous strand $\bar{S}(P,Q)$ of $\mathbb{S}$ is

$$0 \to A(P,Q,0) \to A(P-1,Q-1,1) \to \cdots \to A(P-Q,0,Q) \to 0,$$

if $Q \le P$; and

$$0 \to A(P,Q,0) \to A(P-1,Q-1,1) \to \cdots \to K(0,1,Q-1) \to 0,$$

if $Q = P+1$. These homogeneous strands $\bar{S}(P,Q)$ are exact because the complexes

$$\begin{aligned} &0 \to D_Q G^* \otimes {\textstyle\bigwedge}^0 G^* \to D_{Q-1} G^* \otimes {\textstyle\bigwedge}^1 G^* \to \cdots \to D_0 G^* \otimes {\textstyle\bigwedge}^Q G^* \to 0, \text{ and} \\ &0 \to D_Q G^* \otimes {\textstyle\bigwedge}^0 G^* \to D_{Q-1} G^* \otimes {\textstyle\bigwedge}^1 G^* \to \cdots \to K(0,1,Q-1) \to 0 \end{aligned}$$

are split exact since $1 \le Q$. We conclude that $\mathbb{S} \subseteq M + \boldsymbol{D}(M)$ and that $(\mathbb{S}, \boldsymbol{D})$ is a is split exact complex. $\square$

4. Filtrations.

On numerous occasions we consider a filtration on a complex. We are particularly interested in the associated graded object of the filtration, and for that reason we highlight the ultimate associated graded object, even as we set up the filtration.

Notation. Let $(\mathbb{E}, \boldsymbol{d})$ be a complex and Π be a partially ordered set. Suppose that, as a graded module, $\mathbb{E} = \bigoplus_{p\in\Pi} \mathbb{E}^{[p]}$ and that

$$\text{for each fixed } p \in \Pi, \text{ the modules and maps } \Big(\bigoplus_{p'\le p} \mathbb{E}^{[p']}, \boldsymbol{d}\Big) \text{ form a subcomplex of } (\mathbb{E}, \boldsymbol{d}). \tag{4.1}$$

For each fixed $p \in \Pi$, let $(\mathbb{E}^{[p]}, \boldsymbol{\partial})$ be the quotient complex which is given by the following short exact sequence of complexes:

$$0 \to \Big(\bigoplus_{p'<p} \mathbb{E}^{[p']}, \boldsymbol{d}\Big) \to \Big(\bigoplus_{p'\le p} \mathbb{E}^{[p']}, \boldsymbol{d}\Big) \to (\mathbb{E}^{[p]}, \boldsymbol{\partial}) \to 0.$$

In particular, the map $\boldsymbol{\partial}_i\colon \mathbb{E}_i^{[p]} \to \mathbb{E}_{i-1}^{[p]}$ is equal to the composition

$$\mathbb{E}_i^{[p]} \xrightarrow{\text{incl}} \mathbb{E}_i \xrightarrow{\boldsymbol{d}_i} \mathbb{E}_{i-1} \xrightarrow{\text{proj}} \mathbb{E}_{i-1}^{[p]},$$

and we refer to $\boldsymbol{\partial}$ as the homogeneous part of $\boldsymbol{d}$ of degree zero with respect to Π. We refer to each complex $(\mathbb{E}^{[p]}, \boldsymbol{\partial})$ as a homogeneous strand of the original complex $(\mathbb{E}, \boldsymbol{d})$. The graded complex associated to the above filtration of $\mathbb{E}$ is denoted by $(\mathbb{E}, \boldsymbol{\partial})$ and is equal to $\bigoplus\limits_{p\in\Pi} (\mathbb{E}^{[p]}, \boldsymbol{\partial})$.

We apply the filtration technique in three settings. Proposition 4.2 (a) is a quick proof of the well-known fact that if the associated graded complex is exact, then so is the original complex. Proposition 4.2 (b) will be used to split an acyclic summand from a complex. We can look at one homogeneous strand at a time to determine that $\operatorname{im} \boldsymbol{\partial}_j$ is a summand of $\mathbb{E}_{j-1}$. Proposition 4.2 (b) allows us to conclude that the image of the original map $\boldsymbol{d}_j$ is also a summand of $\mathbb{E}_{j-1}$.

Proposition 4.2. *Let $(\mathbb{E}, \boldsymbol{d})$ be a complex of finitely generated projective R-modules and Π be a partially ordered set. Suppose that $\mathbb{E}$ may be decomposed as a direct sum $\bigoplus_{p\in\Pi} \mathbb{E}^{[p]}$ and that this decomposition satisfies hypothesis 4.1. Fix an integer j.*

(a) *If $\mathrm{H}_j(\mathbb{E}, \boldsymbol{\partial}) = 0$, then $\mathrm{H}_j(\mathbb{E}, \boldsymbol{d}) = 0$.*
(b) *If $\mathrm{H}_j(\mathbb{E}, \boldsymbol{\partial}) = 0$ and $\operatorname{im} \boldsymbol{\partial}_j$ is a summand of $\mathbb{E}_{j-1}$, then $\operatorname{im} \boldsymbol{d}_j$ is also a summand of $\mathbb{E}_{j-1}$.*

Proof. Let x be a non-zero j-cycle of $\mathbb{E}$. Consider $x = \sum x^{[p]}$, with $x^{[p]} \in \mathbb{E}^{[p]}$, and let

$$U(x) = \{p \in \Pi \mid x^{[\pi]} = 0 \text{ for all } \pi \in \Pi \text{ with } p \le \pi\}.$$

Let p_0 be a maximal element of the support of x. It is clear that $\boldsymbol{\partial}(x^{[p_0]}) = 0$. It follows that there exists $y \in \mathbb{E}^{[p_0]}$ with $\boldsymbol{\partial}(y) = x^{[p_0]}$. We see that $U(x) \subsetneq U(x - \boldsymbol{d}y)$. The proof of (a) is completed by induction. We prove (b). Let $\mathbb{E}'_{j-1}$ be a direct sum complement of $\operatorname{im} \boldsymbol{\partial}_j$ in $\mathbb{E}_{j-1}$. Assertion (a) may be applied to

$$\overline{\mathbb{E}}: \quad \mathbb{E}_{j+1} \to \mathbb{E}_j \to \frac{\mathbb{E}_{j-1}}{\mathbb{E}'_{j-1}} \to 0.$$

We are given that $(\overline{\mathbb{E}}, \boldsymbol{\partial})$ is exact. We conclude that $(\overline{\mathbb{E}}, \boldsymbol{d})$ is exact. It follows readily that $\mathbb{E}_{j-1} = \operatorname{im} \boldsymbol{d}_j \oplus \mathbb{E}'_{j-1}$. $\square$

In [16], we said that the complex $\mathbb{L}$ is *splittable* if $\mathbb{L}$ is the direct sum of two subcomplexes $\mathbb{L}'$ and $\mathbb{L}''$, with $\mathbb{L}'$ split exact, and the differential on $\mathbb{L}''$ identically zero. Suppose that $\mathbb{L}$ is a complex of projective modules and $\mathbb{L}$ is bounded in the sense that there exists an integer N with $\mathbb{L}_i = 0$ for all $i < N$. Under these hypotheses, we proved that $\mathbb{L}$ is splittable if and only if $\mathrm{H}_j(\mathbb{L})$ is projective for all j.

Our third application of the filtration technique is stated in Observation 4.3. After the notation is set, then the hypothesis is that various homogeneous strands of the complex $\mathbb{E}$ have been identified and each of these strands contains a splittable substrand. The conclusion is that, in the original non-homogeneous complex $\mathbb{E}$, each splittable substrand may be replaced by its homology, at the expense of complicating the differential.

Observation 4.3. *Let $(\mathbb{E}, \boldsymbol{d})$ be a complex of finitely generated projective R-modules and Π be a partially ordered set. Suppose that $\mathbb{E}$ may be decomposed as a direct sum $\bigoplus_{p\in\Pi} \mathbb{E}^{[p]}$ and that this decomposition satisfies hypothesis 4.1. Suppose that each module $\mathbb{E}_i^{[p]}$ decomposes into $\mathbb{L}_i^{[p]} \oplus \mathbb{K}_i^{[p]}$. Let $\mathbb{L}_i = \bigoplus_p \mathbb{L}_i^{[p]}$, $\mathbb{K}_i = \bigoplus_p \mathbb{K}_i^{[p]}$, and $\mathbb{L} = \bigoplus_i \mathbb{L}_i$. View $\mathbb{L}$ as a substrand of $(\mathbb{E}, \boldsymbol{\partial})$. If $\mathbb{L}$ is a splittable complex, then there exists a split exact subcomplex $(\mathbb{N}, \boldsymbol{d})$ of $(\mathbb{E}, \boldsymbol{d})$ such that $\mathbb{N}$ is a direct summand of $\mathbb{E}$ as a module, and $(\mathbb{E}/\mathbb{N})_i \cong \mathrm{H}_i(\mathbb{L}) \oplus \mathbb{K}_i$.*

Remark. We emphasize that "view $\mathbb{L}$ as a substrand of $(\mathbb{E}, \boldsymbol{\partial})$" means that the differential on $\mathbb{L}$ is

$$\mathbb{L}_i \xrightarrow{\text{incl}} \mathbb{E}_i \xrightarrow{\boldsymbol{\partial}} \mathbb{E}_{i-1} \xrightarrow{\text{proj}} \mathbb{L}_{i-1}.$$

In practice, for a particular choice of i and p, one usually takes either $\mathbb{L}_i^{[p]}$ or $\mathbb{K}_i^{[p]}$ to be zero. When this practice is in effect, then $\mathbb{L}$ is easily seen to be a complex.

Proof. The hypothesis guarantees that $\mathbb{L}$ decomposes into the direct sum of two subcomplexes $\mathbb{P} \oplus \mathbb{Q}$, where $\mathbb{Q}$ is split exact and $\mathbb{P} \cong \mathrm{H}(\mathbb{L})$. For each i, let $\mathbb{Q}_i$ equal $A_i \oplus B_i$, where B_i is equal to the image of $\mathbb{Q}_{i+1}$ in $\mathbb{L}$. We see that the differential in $\mathbb{L}$ carries A_i isomorphically onto B_{i-1}. Observe that $\mathbb{E}_i = A_i \oplus B_i \oplus \mathbb{P}_i \oplus \mathbb{K}_i$ and that the composition

$$A_i \xrightarrow{\text{incl}} \mathbb{E}_i \xrightarrow{\boldsymbol{d}_i} \mathbb{E}_{i-1} \xrightarrow{\text{proj}} B_{i-1} \tag{4.4}$$

is an isomorphism for each i. The second assertion holds because the homogeneous part of (4.4) is an isomorphism. Define $\mathbb{N}$ to be $\bigoplus_i \mathbb{N}_i$ and $\mathbb{M}$ to be $\bigoplus_i \mathbb{M}_i$, with $\mathbb{N}_i = A_i + \boldsymbol{d}_{i+1}(A_{i+1})$ and $\mathbb{M}_i = \mathbb{P}_i \oplus \mathbb{K}_i$. Use the decomposition $\mathbb{E}_i = A_i \oplus B_i \oplus \mathbb{M}_i$ to produce the projection maps

$$\pi_i^B \colon \mathbb{E}_i \to B_i \quad \text{and} \quad \pi_i^{\mathbb{M}} \colon \mathbb{E}_i \to \mathbb{M}_i.$$

Let $\theta_{i-1} \colon B_{i-1} \to A_i$ be the inverse of the map of (4.4); $\psi_i \colon \mathbb{E}_i \to \mathbb{M}_i$ be

$$\psi_i = \pi_i^{\mathbb{M}} \circ (1 - \boldsymbol{d}_{i+1} \circ \theta_i \circ \pi_i^B);$$

and $m_i\colon \mathbb{M}_i \to \mathbb{M}_{i-1}$ be the composition

$$\mathbb{M}_i \xrightarrow{\text{incl}} \mathbb{E}_i \xrightarrow{\boldsymbol{d}_i} \mathbb{E}_{i-1} \xrightarrow{\psi_{i-1}} \mathbb{M}_{i-1}.$$

A straightforward calculation (see, for example, [15, Prop. 7.2] or [14, Prop. 3.14]) shows that

$$0 \to (\mathbb{N}, \boldsymbol{d}|_{\mathbb{N}}) \xrightarrow{\text{incl}} (\mathbb{E}, \boldsymbol{d}) \xrightarrow{\psi} (\mathbb{M}, m) \to 0$$

is a short exact sequence of complexes, and that $\mathbb{N}$ fulfills all of the requirements. We notice, for future reference, that the decomposition $\mathbb{M} = \bigoplus_{p\in\Pi} \mathbb{M}^{[p]}$ also satisfies hypothesis 4.1. $\square$

5. Split a huge summand from $\mathbb{F}$.

In Corollary 5.3 we exhibit a finite free subcomplex $\mathbb{G}$ of $\mathbb{F}$ which has the same homology as $\mathbb{F}$.

Fix the complexes $(\mathbb{F}, \boldsymbol{d})$ and $(\mathbb{F}, \boldsymbol{\partial})$ of Definition 2.4 and 2.9. We define complexes $(\mathbb{P}(a_0, c_0, d_0), \boldsymbol{\partial})$ and $(\mathbb{E}, \boldsymbol{\partial})$. Each of the new complexes is a quotient of $(\mathbb{F}, \boldsymbol{\partial})$ under the natural quotient map. In particular, $\mathbb{P}_i$ and $\mathbb{E}_i$ are defined for all integers i. If we don't specify a value for one of these modules, then the module is automatically equal to zero. The position of the module $A(a, c, d)$ is $a+c+d+1$ in every complex which contains it. Let

$$\bar{A}(a, c, d) = \frac{A(a, c, d)}{\boldsymbol{\partial}(A(a+1, c+1, d-1))} \quad \text{and} \quad \bar{B}(d) = \frac{B(d)}{\boldsymbol{\partial}(A(g, 0, d-g))}.$$

Definition. If a_0, c_0, and d_0 are integers, with a_0 and c_0 non-negative, then let $(\mathbb{P}(a_0, c_0, d_0), \boldsymbol{\partial})$ be the complex

$$0 \to A(a_0+d_0, c_0+d_0, 0) \xrightarrow{\boldsymbol{\partial}} \dots \xrightarrow{\boldsymbol{\partial}} A(a_0+1, c_0+1, d_0-1) \xrightarrow{\boldsymbol{\partial}} A(a_0, c_0, d_0) \to 0.$$

If $P = a_0 + d_0$ and $Q = c_0 + d_0$, then the complex $\mathbb{P}(a_0, c_0, d_0)$ is a quotient of the homogeneous strand $(S(P, Q), \boldsymbol{\partial})$ of Observation 2.8. The homogeneous strands $(S(P, Q), \boldsymbol{\partial})$ have been studied extensively, under a slightly different name, in [16]. The exact connection between the two notations is

$$S(P, Q) = \begin{cases} \mathbb{M}(P, Q) \otimes \bigwedge^b F^*, & \text{if } -e \le P - Q \le g-1, \text{ and} \\ \widetilde{\mathbb{M}}(P, Q) \otimes \bigwedge^b F^*, & \text{if } P = Q + g, \end{cases}$$

for $b = P - Q + e$. The differential $\boldsymbol{\partial}$ of $S(P, Q)$ is equal to the tensor product of the differential of $\mathbb{M}(P, Q)$ or $\widetilde{\mathbb{M}}(P, Q)$ with the identity map on $\bigwedge^b F^*$. The following calculations are Corollaries 5.1 and 5.2 of [16].

Theorem 5.1.

(a) *Assume* $1-e \le P-Q \le g-1$. *If either* $eg-g+1 \le Q$ *or* $eg-e+1 \le P$, *then* $S(P,Q)$ *is split exact.*

(b) *If* $Q = e+P$, *then* $S(P,Q)$ *has free homology which is equal to* $\bar{A}(0,e,P)$; *furthermore, if* $eg-e+1 \le P$, *then* $\bar{A}(0,e,P)=0$.

(c) *If* $g+Q=P$, *then* $S(P,Q)$ *has free homology which is equal to* $\bar{B}(P)$; *furthermore, if* $eg-e+1 \le P$, *then* $\bar{B}(P)=0$.

(d) *Fix integers* a, c, *and* d. *Assume that* $1-e \le a-c \le g-1$. *If* $g-1 \le a$ *or* $e-1 \le c$, *then the complex* $\mathbb{P}(a,c,d)$ *has free homology equal to* $\bar{A}(a,c,d)$.

Lemma 5.2. *If* $(\mathbb{E}, \boldsymbol{\partial})$ *is the complex* $0 \to \bigoplus A(a,c,d) \to 0$, *where the parameters satisfy* $eg \le a+c+d$, *then* $\mathbb{E}$ *is a splittable complex and* $\mathrm{H}(\mathbb{E})$ *is equal to the free module* $\bigoplus \bar{A}(a,c,d)$, *where the parameters satisfy* $eg = a+c+d$.

Proof. Observe that $\mathbb{E}$ is equal to the direct sum

$$\bigoplus_{eg=a_0+c_0+d_0} \mathbb{P}(a_0,c_0,d_0) \oplus \bigoplus_{\substack{eg+1 \le a_0+c_0+d_0 \\ 0=a_0c_0}} \mathbb{P}(a_0,c_0,d_0),$$

with the parameters a_0, c_0, and d_0 all non-negative. Let $\mathbb{P}$ be the strand $\mathbb{P}(a_0,c_0,d_0)$ of $\mathbb{E}$, and let $P = a_0+d_0$. If $eg+1 \le a_0+c_0+d_0$ and $a_0c_0=0$, then Theorem 5.1 yields that $\mathbb{P}$ is split exact. If $a_0 = g+c_0$, then c_0 must be zero; hence, $eg+1 \le P$ and part (c) applies. If $c_0 = a_0+e$, then a_0 must be zero; hence, $eg-e+1 \le P$ and (b) applies. If $1-g \le c_0-a_0 \le e-1$, then $c_0 \le e-1$; hence, $eg-e+1 \le P$ and (a) applies. If $a_0+c_0+d_0 = eg$, then $\mathbb{P}$ has free homology equal to $\bar{A}(a_0,c_0,d_0)$. Indeed, either (d) applies directly or else (a)–(c) yield that $\mathbb{P}$ is a truncation of a split exact complex of free modules. In the later case the homology of $\mathbb{P}(a_0,c_0,d_0)$ is the projective module $\bar{A}(a_0,c_0,d_0)$. All of the complexes are made over $\mathbb{Z}$ and transfered to the arbitrary ring R of 2.1 by way of base change. In any event $\bar{A}(a_0,c_0,d_0)$ is a free module. $\square$

Corollary 5.3. *Let* $(\mathbb{F}, \boldsymbol{d})$ *be the complex of Definition 2.4. If* $A(a,c,d)$ *is a summand of* $\mathbb{F}$ *with* $a+c+d = eg$, *then there exists a free submodule* $A'(a,c,d)$ *of* $A(a,c,d)$ *so that*

$$A(a,c,d) = A'(a,c,d) \oplus \boldsymbol{\partial} A(a+1,c+1,d-1).$$

If $\mathbb{G}$ *is the submodule*

$$\bigoplus_{a+c+d=eg} A'(a,c,d) \oplus \bigoplus_{a+c+d \le eg-1} A(a,c,d) \oplus \bigoplus_i B(i)$$

of $\mathbb{F}$, *then* $\mathrm{H}_i(\mathbb{G}, \boldsymbol{d}) = \mathrm{H}_i(\mathbb{F}, \boldsymbol{d})$ *for all integers* i.

Proof. Lemma 5.2 guarantees the existence of $A'(a,c,d)$. Apply Proposition 4.2 to see that $\mathbb{F}_{eg+1} = \operatorname{im} \boldsymbol{d}_{eg+2} \oplus \bigoplus_{a+c+d=eg} A'(a,c,d)$ and that the cokernel of the inclusion $\mathbb{G} \subseteq \mathbb{F}$ is split exact. $\square$

6. The complex $\mathbb{F}$ is acyclic.

Theorem 6.1. *If the data $(\mathfrak{b}, V, X)$ of 2.1 is generic, then the complex $(\mathbb{F}, \boldsymbol{d})$ of Definition 2.4 is acyclic.*

Proof. Corollary 5.3 ensures the existence of a subcomplex $\mathbb{G}$ of $\mathbb{F}$ such that $\mathbb{G}$ consists of free modules, $\mathbb{G}$ has length $eg+1$, and $\mathbb{F}/\mathbb{G}$ is split exact. According to the acyclicity lemma [7, Cor. 4.2], it suffices to show that $\mathbb{G}_P$ is acyclic for all prime ideals P of R with $\operatorname{grade} P < eg+1$. Thus, it suffices to show that $\mathbb{F}_P$ is acyclic for all prime ideals P of R with $\operatorname{grade} P < eg+1$. The ideal $I_1(V)$ has grade $ef \geq eg+1$. If v is an entry of a matrix representation of V, then Lemma 6.2 shows that $\mathbb{F}_v$ is isomorphic to the complex $\mathbb{F}$ of Lemma 6.3, which is built with generic data over the ring $R_0[v^{-1}]$. The complex $\mathbb{F}$ of Lemma 6.3 has the same homology as $\mathbb{F}' \otimes \mathbb{K}$, where $\mathbb{F}'$ is made from generic data, with $e-1$ in place of e, and $\mathbb{K}$ is the Koszul complex associated to the sequence of g new indeterminates. Induction on e completes the result. The base case is Theorem 3.1. $\square$

Lemma 6.2. *Form the complex $(\mathbb{F}, \boldsymbol{d})$ using the data $(\mathfrak{b}, V, X)$ of 2.1. Let θ and τ be automorphisms of F and E, respectively. Form $(\mathbb{F}, \boldsymbol{d}')$ using $(u'\mathfrak{b}, \theta \circ V, X \circ \theta^{-1})$ and form $(\mathbb{F}, \boldsymbol{d}'')$ using $(u''\mathfrak{b}, V \circ \tau, X)$, where $u' = ((\bigwedge^f \theta^{-1})[\omega_F])(\omega_{F^*})$ and u'' is $[(\bigwedge^e \tau^{-1})(\omega_E)](\omega_{E^*})$. Then the complexes $(\mathbb{F}, \boldsymbol{d})$, $(\mathbb{F}, \boldsymbol{d}')$, and $(\mathbb{F}, \boldsymbol{d}'')$ are isomorphic to one another.*

Proof. Define $\varphi' \colon (\mathbb{F}, \boldsymbol{d}) \to (\mathbb{F}, \boldsymbol{d}')$ and $\varphi'' \colon (\mathbb{F}, \boldsymbol{d}) \to (\mathbb{F}, \boldsymbol{d}'')$, by $\varphi'(z_d) = u'z_d$ in $B(d)$, $\varphi''(z_d) = (\bigwedge^d(\tau^{-1} \otimes 1))(z_d) \in B(d)$,

$$\begin{aligned}
&\varphi'(x) = \varepsilon_1^{(a)} \otimes \gamma_1^{(c)} \otimes z_d \otimes (\textstyle\bigwedge^b \theta^{*-1})(\alpha_b) \in A(a,c,d), \text{ and} \\
&\varphi''(x) = (D_a \tau^{-1})(\varepsilon_1^{(a)}) \otimes \gamma_1^{(c)} \otimes (\textstyle\bigwedge^d(\tau^{-1} \otimes 1))(z_d) \otimes \alpha_b \in A(a,c,d),
\end{aligned}$$

for $z_d \in B(d)$, and $x = \varepsilon_1^{(a)} \otimes \gamma_1^{(c)} \otimes z_d \otimes \alpha_b \in A(a,c,d)$. It is not difficult to see that φ' and φ'' both are isomorphisms of complexes. $\square$

Lemma 6.3. *Let $(\mathfrak{b}, V, X)$ be the data of 2.1. Suppose that $E = E' \oplus E''$ and $F = F' \oplus F''$, with $E'' = R\varepsilon$ and $F'' = Rf$. Suppose further that*

$$V = \begin{bmatrix} V' & 0 \\ 0 & V'' \end{bmatrix}, \quad \textit{and} \quad X = [\, X' \quad X'' \,], \quad \textit{where}$$

$$E'' \xrightarrow{V''} F'', \quad E' \xrightarrow{V'} F' \xrightarrow{X'} G, \quad \textit{and} \quad F'' \xrightarrow{X''} G$$

are R-module homomorphisms, and $V''(\varepsilon) = f$. Form the complexes $(\mathbb{F}, \boldsymbol{d})$ and $(\mathbb{F}', \boldsymbol{d}')$ using the data $(\mathfrak{b}, V, X)$ and $(\mathfrak{b}, V', X')$, respectively. Let $x'' = X''(f) \in G$. Then there exists a split exact complex $\mathbb{L}$ and a short exact sequence of complexes:

$$0 \to (\mathbb{F}', \boldsymbol{d}') \otimes (\textstyle\bigwedge^\bullet G^*, x'') \to \mathbb{F} \to \mathbb{L} \to 0.$$

Proof. Take $\omega_E = \varepsilon \wedge \omega_{E'}$ and $\omega_F = f \wedge \omega_{F'}$. Let α be the element of F^* with $\alpha(F') = 0$ and $\alpha(f) = 1$, $A(a_1, c, d_1; a_2, d_2; b_1, b_2)$ equal

$$D_{a_1}E' \otimes D_{a_2}E'' \otimes D_cG^* \otimes \textstyle\bigwedge^{d_1}(E' \otimes G^*) \otimes \bigwedge^{d_2}(E'' \otimes G^*) \otimes \bigwedge^{b_1}F'^* \otimes \bigwedge^{b_2}F''^*,$$

and $B(d_1; d_2) = \bigwedge^{d_1}(E' \otimes G^*) \otimes \bigwedge^{d_2}(E'' \otimes G^*)$. We see that

$$A(a, c, d) = \sum A(a_1, c, d_1; a_2, d_2; b_1, b_2) \quad \text{and} \quad B(d) = \sum B(d_1; d_2),$$

where the first sum varies over all tuples $(a_1, a_2, d_1, d_2, b_1, b_2)$ with $a_1 + a_2 = a$, $b_1 + b_2 = a - c + e$, and $d_1 + d_2 = d$, and the second sum varies over all tuples (d_1, d_2) with $d_1 + d_2 = d$. The complex $(\mathbb{F}, d)$ is built using the modules $A(a, c, d)$ and $B(d)$. The complex $(\mathbb{F}', \boldsymbol{d}')$ is built using the modules

$$A'(a, c, d) = A(a, c, d; 0, 0; b_1, 0) \quad \text{and} \quad B'(d) = B(d; 0),$$

where $b_1 = a - c + e - 1$. The differential in the complex $\Big((\mathbb{F}', \boldsymbol{d}') \otimes (\bigwedge^\bullet G^*, x''), D\Big)$ is given by

$$D(a \otimes b) = \boldsymbol{d}'(a) \otimes b + (-1)^{|a|+1} a \otimes x''(b).$$

Define $\varphi : \Big((\mathbb{F}', \boldsymbol{d}') \otimes (\bigwedge^\bullet G^*, x''), D\Big) \to (\mathbb{F}, \boldsymbol{d})$ by

$$\varphi\left(z_{d_1} \otimes \gamma_{d_2} \in B'(d_1) \otimes \textstyle\bigwedge^{d_2} G^*\right) = (-1)^{d_2} z_{d_1} \wedge (\varepsilon^{(d_2)} \bowtie \gamma_{d_2}) \in B(d_1; d_2),$$

and $\varphi\left(\varepsilon_1^{(a_1)} \otimes \gamma_1^{(c)} \otimes z_{d_1} \otimes \alpha_{b_1} \otimes \gamma_{d_2} \in A'(a_1, c, d_1) \otimes \bigwedge^{d_2} G^*\right)$ is equal to

$$\varepsilon_1^{(a_1)} \otimes \gamma_1^{(c)} \otimes z_{d_1} \wedge (\varepsilon^{(d_2)} \bowtie \gamma_{d_2}) \otimes \alpha_{b_1} \wedge \alpha \in A(a_1, c, d_1; 0, d_2; b_1, 1).$$

It is clear that φ is injective. We see that φ is a map of complexes, because $\boldsymbol{d} \circ \varphi$ and $\varphi \circ D$ both carry the element $z_{d_1} \otimes \gamma_{d_2}$ of $B'(d_1) \otimes \bigwedge^{d_2} G^*$ to

$$(-1)^{d_2}(X' \check{\circ} V')(z_{d_1}) \wedge (\varepsilon^{(d_2)} \bowtie \gamma_{d_2}) + (-1)^{d_1 + d_2} z_{d_1} \wedge \left(\varepsilon^{(d_2 - 1)} \bowtie x''(\gamma_{d_2})\right),$$

and carry the element $\varepsilon_1^{(a_1)} \otimes \gamma_1^{(c)} \otimes z_{d_1} \otimes \alpha_{b_1} \otimes \gamma_{d_2}$ of $A'(a_1, c, d_1) \otimes \bigwedge^{d_2} G^*$ to

$$\begin{cases} \varepsilon_1^{(a_1-1)} \otimes \gamma_1^{(c)} \otimes z_{d_1} \wedge (\varepsilon^{(d_2)} \bowtie \gamma_{d_2}) \otimes [V'(\varepsilon_1)](\alpha_{b_1}) \wedge \alpha \\ -\varepsilon_1^{(a_1)} \otimes \gamma_1^{(c-1)} \otimes z_{d_1} \wedge (\varepsilon^{(d_2)} \bowtie \gamma_{d_2}) \otimes X'^*(\gamma_1) \wedge \alpha_{b_1} \wedge \alpha \\ +(-1)^{a_1+c} \varepsilon_1^{(a_1)} \otimes \gamma_1^{(c)} \otimes (X' \check{\circ} V')(z_{d_1}) \wedge (\varepsilon^{(d_2)} \bowtie \gamma_{d_2}) \otimes \alpha_{b_1} \wedge \alpha \\ +(-1)^{a_1+c+d_1} \varepsilon_1^{(a_1)} \otimes \gamma_1^{(c)} \otimes z_{d_1} \wedge (\varepsilon^{(d_2-1)} \bowtie x''(\gamma_{d_2})) \otimes \alpha_{b_1} \wedge \alpha \\ +(-1)^{a_1+c} \varepsilon_1^{(a_1-1)} \otimes \gamma_1^{(c-1)} \otimes (\varepsilon_1 \otimes \gamma_1) \wedge z_{d_1} \wedge (\varepsilon^{(d_2)} \bowtie \gamma_{d_2}) \otimes \alpha_{b_1} \wedge \alpha \\ +(-1)^{a_1+d_1+d_2} \delta_{c0} \varepsilon_1^{(a_1)} \bowtie [(\bigwedge^{f-b_1-1} X')(\alpha_{b_1}[\omega_{F'}])](\omega_{G^*}) \wedge z_{d_1} \wedge (\varepsilon^{(d_2)} \bowtie \gamma_{d_2}) \\ +(-1)^{d_1+d_2} \chi(a_1 = 0) \mathfrak{b} \cdot [(\bigwedge^{b_1} V'^*)(\alpha_{b_1})](\omega_{E'}) \bowtie \gamma_1^{(c)} \wedge z_{d_1} \wedge (\varepsilon^{(d_2)} \bowtie \gamma_{d_2}). \end{cases}$$

The cokernel of φ is the direct sum of all $A(a_1, c, d_1; a_2, d_2; b_1, b_2)$ such that either $0 < a_2$ and $1 = b_2$; or $0 \le a_2$ and $0 = b_2$. Decompose $\operatorname{coker} \varphi$ into a direct sum of strands $\mathcal{S}(P, Q)$, where

$$A(a_1, c, d_1; a_2, d_2; b_1, b_2) \in \mathcal{S}(P, Q) \text{ if } P = a_2 - b_2 + d_2 \quad \text{and} \quad Q = a_1 + c + d_1.$$

Impose the inverse lexicographic order of 2.7 on $\{(P, Q)\}$. It is easy to see that the decomposition $\operatorname{coker} \varphi = \bigoplus \mathcal{S}(P, Q)$ satisfies hypothesis 4.1 and that each strand $(\mathcal{S}(P, Q), \boldsymbol{\partial})$ is equal to the split exact sequence

$$\bigoplus 0 \to A(a_1, c, d_1; a_2, d_2; b_1, 1) \xrightarrow{\cong} A(a_1, c, d_1; a_2 - 1, d_2; b_1, 0) \to 0,$$

where the sum varies over all tuples $(a_1, c, d_1; a_2, d_2; b_1, 1)$ with $a_2 + d_2 = P + 1$, $a_1 + c + d_1 = Q$, and $1 \le a_2$. Apply Proposition 4.2. $\square$

The ideal $\mathcal{J}$ of (0.2) is generically perfect of grade $eg + 1$. The theorem about the transfer of perfection (see, for example, [5, Theorem 3.5]) tells us that $\mathbb{F} \otimes_{\mathcal{P}} R$ and $\mathbb{G} \otimes_{\mathcal{P}} R$ are resolutions for any ring R for which $\mathcal{J} R$ is a proper ideal of grade at least $eg + 1$.

7. The minimal resolution.

Data 7.1. Let $\boldsymbol{K}$ be a field and $\mathcal{R} = \mathcal{P}/\mathcal{J}$ be the universal ring of (0.2). We write $\mathcal{P}'$ for the polynomial ring $\boldsymbol{K} \otimes_{\mathbb{Z}} \mathcal{P} = \boldsymbol{K}[\mathfrak{b}, \{v_{jk}\}, \{x_{ij}\}]$, $\mathcal{J}'$ for the image of $\mathcal{J}$ in $\mathcal{P}'$, and $\mathcal{R}'$ for $\boldsymbol{K} \otimes_{\mathbb{Z}} \mathcal{R} = \mathcal{P}'/\mathcal{J}'$. Let E_0, F_0, and G_0 be vector spaces of dimension e, f, and g over $\boldsymbol{K}$, and $E = E_0 \otimes_{\boldsymbol{K}} \mathcal{P}'$, $F = F_0 \otimes_{\boldsymbol{K}} \mathcal{P}'$, and $G = G_0 \otimes_{\boldsymbol{K}} \mathcal{P}'$ be the corresponding free $\mathcal{P}'$-modules. Let $(\mathbb{F}, \boldsymbol{d})$ be the complex of 2.4 built using the data $(\mathfrak{b}, V, X)$:

$$E \xrightarrow{V} F \xrightarrow{X} G,$$

where $V = [v_{jk}]$ and $X = [x_{ij}]$ are matrices.

In Theorem 7.6, we record the modules of the minimal $\mathcal{A}$-homogeneous resolution of $\mathcal{R}'$ by free $\mathcal{P}'$-modules. There are two steps in our proof of Theorem 7.6. In the first step, Lemma 7.2, we apply the technique of Observation 4.3 to the present situation. The other step is the calculation of the homology of the homogeneous strands of $\mathbb{F}$. This step was largely carried out in [16]. Most of the modules that comprise the resolution of 7.6 are equal to modules which arise when one resolves divisors of a determinantal ring defined by the 2×2 minors of an $e \times g$ matrix.

Lemma 7.2. *Adopt the hypotheses of 7.1. If* $\mathbb{X}$ *is the minimal* $\mathcal{A}$*-homogeneous resolution of* $\mathcal{R}'$ *as a* $\mathcal{P}'$*-module, then* $\mathbb{X}$ *and* $\mathrm{H}(\mathbb{F} \otimes_{\mathcal{P}'} \boldsymbol{K}) \otimes_{\boldsymbol{K}} \mathcal{P}'$ *are isomorphic as* $\mathcal{A}$*-graded* $\mathcal{P}'$*-modules.*

Proof. The homology of $\mathbb{F} \otimes_{\mathcal{P}'} \boldsymbol{K}$ is free over $\boldsymbol{K}$, since $\boldsymbol{K}$ is a field; and therefore, $\mathbb{F} \otimes_{\mathcal{P}'} \boldsymbol{K}$ is equal to the direct sum of two graded subcomplexes, one of which is split exact and the other has zero differential. Use the graded version of Nakayama's Lemma to pull this decomposition back to $\mathbb{F}$. At this point, the summand $\mathbb{F}_i$ of $\mathbb{F}$ has been decomposed as the direct sum $A_i \oplus B_i \oplus C_i$ of graded free $\mathcal{P}'$-modules and the graded differential $\boldsymbol{d}_i \colon \mathbb{F}_i \to \mathbb{F}_{i-1}$, which looks like

$$\begin{bmatrix} \boldsymbol{d}_i^{11} & \boldsymbol{d}_i^{12} & \boldsymbol{d}_i^{13} \\ \boldsymbol{d}_i^{21} & \boldsymbol{d}_i^{22} & \boldsymbol{d}_i^{23} \\ \boldsymbol{d}_i^{31} & \boldsymbol{d}_i^{32} & \boldsymbol{d}_i^{33} \end{bmatrix},$$

has $\boldsymbol{d}_i^{13} \otimes_{\mathcal{P}'} \boldsymbol{K} \colon C_i \otimes_{\mathcal{P}'} \boldsymbol{K} \to A_{i-1} \otimes_{\mathcal{P}'} \boldsymbol{K}$ is an isomorphism and $\boldsymbol{d}_i^{k\ell} \otimes_{\mathcal{P}'} \boldsymbol{K} = 0$ if $(k, \ell) \neq (1, 3)$. It follows that the map $\boldsymbol{d}_i^{13} \colon C_i \to A_{i-1}$, of graded free $\mathcal{P}'$-modules, is also an isomorphism. Let $\mathbb{C}$ be the subcomplex $\bigoplus_i C_i + \boldsymbol{d}(\bigoplus_i C_i)$ of $\mathbb{F}$. Ordinary row and column operations produce a short exact sequence of complexes of graded free $\mathcal{P}'$-modules

$$0 \to (\mathbb{C}, \boldsymbol{d}) \xrightarrow{\text{incl}} (\mathbb{F}, \boldsymbol{d}) \to (\mathbb{X}, x) \to 0,$$

where $\mathbb{X} = \bigoplus_i B_i$, $x_i \colon B_i \to B_{i-1}$ is $\boldsymbol{d}_i^{22} - \boldsymbol{d}_i^{23} \circ (\boldsymbol{d}_i^{13})^{-1} \circ \boldsymbol{d}_i^{12}$, and $(\mathbb{F}, \boldsymbol{d}) \to (\mathbb{X}, x)$ induces an isomorphism on homology. The explicit form of the map x_i guarantees that the differential in $\mathbb{X} \otimes_{\mathcal{P}'} \boldsymbol{K}$ is zero. $\square$

Now we must identify the homology of $\mathbb{F} \otimes_{\mathcal{P}'} \boldsymbol{K}$. We begin by recalling the bi-graded structure on Tor.

Definition. If $\mathfrak{P} = \bigoplus_i \mathfrak{P}_i$ is a graded ring, and $A = \bigoplus_i A_i$ and $B = \bigoplus_i B_i$ are graded $\mathfrak{P}$-modules, then the module $\mathrm{Tor}_\bullet^{\mathfrak{P}}(A, B)$ is a bi-graded $\mathfrak{P}$-module. Indeed, if

$$\mathbb{Y} \colon \cdots \to Y_1 \to Y_0 \to A$$

is a $\mathfrak{P}$-free resolution of A, homogeneous of degree zero, then

$$\operatorname{Tor}^{\mathfrak{P}}_{p,q}(A,B)=\frac{\ker[(Y_p\otimes B)_q\to(Y_{p-1}\otimes B)_q]}{\operatorname{im}[(Y_{p+1}\otimes B)_q\to(Y_p\otimes B)_q]}.$$

Notation 7.3. Adopt the notation of 7.1. Define the $\boldsymbol{K}$-vector spaces

$$\begin{aligned}\mathcal{N}(a,c,d)&=S_aE_0^*\otimes S_cG_0\otimes\textstyle\bigwedge^d(E_0^*\otimes G_0),\\ \mathcal{M}(a,c,d)&=D_aE_0\otimes D_cG_0^*\otimes\textstyle\bigwedge^d(E_0\otimes G_0^*),\text{ and}\\ B_0(i)&=\textstyle\bigwedge^i(E_0\otimes_{\boldsymbol{K}}G_0^*).\end{aligned}$$

The identity map on $E_0^*\otimes G_0$ induces Koszul complexes of the form

$$\cdots\to\mathcal{N}(a-1,c-1,d+1)\to\mathcal{N}(a,c,d)\to\mathcal{N}(a+1,c+1,d-1)\to\cdots\ . \tag{0.7}$$

The R-dual of (0.7) is

$$\cdots\to\mathcal{M}(a+1,c+1,d-1)\to\mathcal{M}(a,c,d)\to\mathcal{M}(a-1,c-1,d+1)\to\cdots\ . \tag{0.8}$$

Fix integers P and Q. Let $\mathbb{N}(P,Q)$ and $\mathbb{M}(P,Q)$ be the above complexes when $a+d=P$ and $c+d=Q$; that is, $\mathbb{N}(P,Q)$ is

$$0\to\mathcal{N}(P-eg,Q-eg,eg)\to\ldots\to\mathcal{N}(P-1,Q-1,1)\to\mathcal{N}(P,Q,0)\to0,$$

and $\mathbb{M}(P,Q)$ is

$$0\to\mathcal{M}(P,Q,0)\to\mathcal{M}(P-1,Q-1,1)\to\ldots\to\mathcal{M}(P-eg,Q-eg,eg)\to0.$$

If $P=g+Q$, then let $\widetilde{\mathbb{M}}(g+Q,Q)$ be the augmented complex

$$0\to\mathcal{M}(g+Q,Q,0)\to\ldots\to\mathcal{M}(g,0,Q)\xrightarrow{\gamma}B_0(g+Q),$$

where $\gamma(U\otimes1\otimes Z)=(U\bowtie\omega_{G^*})\wedge Z$.

Remark. In [16], the homology of (0.7) at $\mathcal{N}(a,c,d)$ is denoted by $\mathrm{H}_{\mathcal{N}}(a,c,d)$ and the cohomology of (0.8) at $\mathcal{M}(a,c,d)$ is called $\mathrm{H}_{\mathcal{M}}(a,c,d)$.

Recall the strands $(S(P,Q),\boldsymbol{\partial})$ of $\mathbb{F}$, which were introduced at the end of section 2. Observe that

$$S(P,Q)\otimes_{\mathcal{P}'}\boldsymbol{K}=\begin{cases}\mathbb{M}(P,Q)\otimes_{\boldsymbol{K}}\bigwedge^{P-Q+e}F_0^* & \text{if } P<Q+g\\ \widetilde{\mathbb{M}}(P,Q)\otimes_{\boldsymbol{K}}\bigwedge^{f}F_0^* & \text{if } P=Q+g.\end{cases} \tag{7.4}$$

Most of the modules that comprise the resolution of 7.6 are equal to modules which arise when one resolves divisors of a determinantal ring defined by 2×2 minors. Let S be the ring $S_\bullet^{\boldsymbol{K}} E_0^* \otimes_{\boldsymbol{K}} S_\bullet^{\boldsymbol{K}} G_0$, T be the subring

$$T = \sum_m S_m E_0^* \otimes S_m G_0$$

of S, and for each integer ℓ, let M_ℓ be the T-submodule

$$M_\ell = \sum_{m-n=\ell} S_m E_0^* \otimes S_n G_0$$

of S. Give S a grading by saying that $S_m E_0^* \otimes S_n G_0$ has grade n, for all m and n. We see that T is a graded ring, and $\bigoplus M_\ell$ is a direct sum decomposition of S into graded T-submodules. In particular,

the graded summand of degree n in M_ℓ is $S_{n+\ell} E_0^* \otimes S_n G_0$.

Let $\mathfrak{P}$ be the polynomial ring $S_\bullet^{\boldsymbol{K}}(E_0^* \otimes_{\boldsymbol{K}} G_0)$. The ring $\mathfrak{P}$ is graded; each element of $S_n(E_0^* \otimes G_0)$ is homogeneous of grade n. The identity map on $E_0^* \otimes G_0$ induces a graded ring homomorphism from $\mathfrak{P}$ onto T. Each graded T-module is automatically a graded $\mathfrak{P}$-module. Notice that $\mathfrak{P} \otimes_{\boldsymbol{K}} \bigwedge_{\boldsymbol{K}}^\bullet (E_0^* \otimes G_0)$ is a homogeneous resolution of $\boldsymbol{K}$ by free $\mathfrak{P}$-modules; and therefore, $\operatorname{Tor}_\bullet^{\mathfrak{P}}(M_\ell, \boldsymbol{K})$ is the homology of $M_\ell \otimes_{\boldsymbol{K}} \bigwedge_{\boldsymbol{K}}^\bullet (E_0^* \otimes G_0)$; indeed,

$$\operatorname{Tor}_{p,q}^{\mathfrak{P}}(M_\ell, \boldsymbol{K}) = \mathrm{H}_{\mathcal{N}}(\ell + q - p, q - p, p).$$

The homology $\operatorname{Tor}_{p,q}^{\mathfrak{P}}(M_\ell, \boldsymbol{K})$ is a $\boldsymbol{K}$-vector space for all p, q, and ℓ. It follows that

$$\mathrm{H}_{\mathcal{M}}(a, c, d) = \mathrm{H}_{\mathcal{N}}(a, c, d) = \operatorname{Tor}_{d,c+d}^{\mathfrak{P}}(M_{a-c}, \boldsymbol{K}) \tag{7.5}$$

for all integers a, c, and d. One may view $\mathfrak{P}$ as a polynomial ring over $\boldsymbol{K}$ in eg indeterminates. The ring T is the determinantal ring defined by the 2×2 minors of the $e \times g$ matrix of indeterminates. The divisor class group of T is $\mathbb{Z}$ and $\ell \mapsto [M_\ell]$ is an isomorphism from $\mathbb{Z}$ to $\mathrm{C}\ell\,(T)$. Much more information about the modules M_ℓ may be found in [4].

We are ready to record the modules in the minimal $\mathcal{P}'$-resolution of $\mathcal{R}'$. Almost all of these modules appear in the minimal $\mathfrak{P}$-resolution of M_ℓ for some ℓ, with $-e \le \ell \le g$. The other type of module that appears in 7.6 is the cokernel of the map γ of (7.3), let $\bar{B}_0(i) = \frac{B_0(i)}{\operatorname{im} \mathcal{M}(g,0,i-g)}$. Write $\boldsymbol{\alpha}$ for $(e-1)(g-1)$.

Theorem 7.6. *Adopt the hypotheses of 7.1. Then the minimal $\mathcal{A}$-homogeneous resolution*

$$\mathbb{X}\colon \quad 0 \to \mathbb{X}_{eg+1} \to \cdots \to \mathbb{X}_0 \to \mathcal{R}' \to 0$$

of $\mathcal{R}'$ by free $\mathcal{P}'$-modules has $\mathbb{X}_i$ equal to

$$\begin{cases} \chi(i \le eg-e)\mathcal{P}' \otimes_{\boldsymbol{K}} \bar{B}_0(i)[-i,-i] \\ \qquad\qquad\oplus \\ \bigoplus\limits_{(p,q,\ell)} \mathcal{P}' \otimes_{\boldsymbol{K}} \left[\operatorname{Tor}^{\mathfrak{P}}_{p,q}(M_\ell, \boldsymbol{K}) \otimes_{\boldsymbol{K}} \bigwedge^{\ell+e} F_0^*\right][-\ell-q,-g-q] \\ \qquad\qquad\oplus \\ \chi(i \le eg+1)\mathcal{P}' \otimes_{\boldsymbol{K}} \operatorname{Tor}^{\mathfrak{P}}_{i-1-e,i-1}(M_{-e}, \boldsymbol{K})[e+1-i, 1-i-g], \end{cases}$$

where the sum is taken over all parameters (p,q,ℓ) with

$$1-e \le \ell \le g-1, \quad 0 \le p \le \boldsymbol{\alpha},$$

$$p + \max\{-\ell, 0\} \le q \le p + \min\{g-1-\ell, e-1\} \quad \textit{and} \quad i = \ell + 2q - p + 1.$$

Proof. We apply Lemma 7.2 to the complex $(\mathbb{F}, \boldsymbol{d})$ of Theorem 6.1. Observation 2.8 shows that $\mathbb{F} \otimes_{\mathcal{P}'} \boldsymbol{K}$ splits into the following direct sum of complexes:

$$\bigoplus_{(P,Q)} (S(P,Q) \otimes_{\mathcal{P}'} \boldsymbol{K}, \boldsymbol{\partial}),$$

where the sum is taken over all integers (P,Q) with $0 \le P$ and $-g \le Q-P \le e$. Apply (7.4) to see that

$$\mathbb{F} \otimes_{\mathcal{P}'} \boldsymbol{K} = \begin{cases} \bigoplus\limits_{P} \left[\widetilde{\mathbb{M}}(P, P-g)\right][-P,-P] \\ \qquad\qquad\oplus \\ \bigoplus\limits_{(P,Q)} \left[\mathbb{M}(P,Q) \otimes_{\boldsymbol{K}} \bigwedge^{P-Q+e} F_0^*\right][-P,-g-Q], \end{cases} \tag{7.7}$$

where the top sum is taken over all integers P, and the bottom sum is taken over all pairs (P,Q) with $-e \le P-Q \le g-1$. It is shown in [16] that the homology of each complex $\widetilde{\mathbb{M}}(P, P-g)$ is free and is concentrated in the position of $B(P)$. It follows immediately that the contribution of the top line of (7.7) to $\mathrm{H}_i(\mathbb{F}\otimes_{\mathcal{P}'}\boldsymbol{K})$ is $\bar{B}_0(i)[-i,-i]$. The contribution of the bottom line of (7.7) to $\mathrm{H}_i(\mathbb{F}\otimes_{\mathcal{P}'}\boldsymbol{K})$ is

$$\bigoplus_{(P,Q)}\left[\mathrm{H}_{\mathcal{M}}(a,c,d)\otimes_{\boldsymbol{K}}\bigwedge^{P-Q+e}F_0^*\right][-P,-g-Q],$$

where (P,Q) continue to satisfy $-e\le P-Q\le g-1$, and the parameters (a,c,d) satisfy $a+d=P$, $c+d=Q$, and $a+c+d+1=i$. Apply (7.5) and reparameterize by letting $\ell=P-Q$, $q=Q$, and $p=d$ to see that

$$\mathbb{X}_i=\begin{cases}\mathcal{P}'\otimes_{\boldsymbol{K}}\bar{B}_0(i)[-i,-i]\\ \qquad\qquad\oplus\\ \bigoplus\limits_{(p,q,\ell)}\mathcal{P}'\otimes_{\boldsymbol{K}}\left[\mathrm{Tor}^{\mathfrak{P}}_{p,q}(M_\ell,\boldsymbol{K})\otimes_{\boldsymbol{K}}\bigwedge^{\ell+e}F_0^*\right][-\ell-q,-g-q],\end{cases}$$

where the sum is taken over all parameters (p,q,ℓ) with

$$i=\ell+2q-p+1\quad\text{and}\quad -e\le\ell\le g-1.$$

We know from [16], that

$$0\to\mathrm{H}_{\mathcal{M}}(g,0,i-g)\to B_0(i)\to\mathrm{H}_{\mathcal{N}}(0,e,eg-e-i)\to 0 \tag{7.8}$$

is an exact sequence. Moreover,

$$\bar{B}_0(i)=\operatorname{coker}\left(\mathrm{H}_{\mathcal{M}}(g,0,i-g)\to B_0(i)\right)=\mathrm{H}_{\mathcal{N}}(0,e,eg-e-i).$$

The module $\mathrm{H}_{\mathcal{N}}(0,e,eg-e-i)$ is zero if $eg-e-i<0$; and therefore,

$$\bar{B}_0(i)=\chi(i\le eg-e)\bar{B}_0(i).$$

Fix $\ell=-e$. The module $\mathrm{Tor}^{\mathfrak{P}}_{p,q}(M_{-e},\boldsymbol{K})$ is equal to $\mathrm{H}_{\mathcal{M}}(q-p-e,q-p,p)$ and [16] tells us that this module is zero unless $q=p+e$. Furthermore, if $q=p+e$, then there is a short exact sequence,

$$0\to\mathrm{H}_{\mathcal{M}}(0,e,p)\to B_0(e+p)\to\mathrm{H}_{\mathcal{N}}(g,0,eg-e-g-p)\to 0.$$

The module $B_0(e+p)$ is zero if $eg<e+p$. If $i=\ell+2q-p+1$ and $\mathrm{Tor}^{\mathfrak{P}}_{p,q}(M_{-e},\boldsymbol{K})\neq 0$, then

$$q=i-1,\quad i-e-1=p,\quad\text{and}\quad i\le eg+1.$$

When $\ell=-e$, the contribution of

$$\bigoplus_{(p,q,\ell)} \mathcal{P}'\otimes_{\boldsymbol{K}}\left[\operatorname{Tor}^{\mathfrak{P}}_{p,q}(M_\ell,\boldsymbol{K})\otimes_{\boldsymbol{K}}\bigwedge\nolimits^{\ell+e}F_0^*\right][-\ell-q,-g-q]$$

to $\mathbb{X}_i$ is

$$\chi(i\le eg+1)\mathcal{P}'\otimes_{\boldsymbol{K}}\operatorname{Tor}^{\mathfrak{P}}_{i-e-1,i-1}(M_{-e},\boldsymbol{K})[e+1-i,-g-i+1].$$

Fix ℓ with $1-e\le\ell\le g-1$. The module

$$\operatorname{Tor}^{\mathfrak{P}}_{p,q}(M_\ell,\boldsymbol{K})=\mathrm{H}_{\mathcal{M}}(q-p+\ell,q-p,p)$$

and [16, Thm. 1.1] tells us that this module is isomorphic to

$$\mathrm{H}_{\mathcal{N}}(g-1-(q-p+\ell),e-1-(q-p),\boldsymbol{\alpha}-p).$$

It follows that $\operatorname{Tor}^{\mathfrak{P}}_{p,q}(M_\ell,\boldsymbol{K})$ is zero unless

$$0\le q-p+\ell\le g-1,\quad 0\le q-p\le e-1,\quad\text{and}\quad 0\le p\le\boldsymbol{\alpha};$$

and therefore, $\operatorname{Tor}^{\mathfrak{P}}_{p,q}(M_\ell,\boldsymbol{K})$ is zero unless

$$p+\max\{-\ell,0\}\le q\le p+\min\{g-1-\ell,e-1\}\quad\text{and}\quad 0\le p\le\boldsymbol{\alpha}.\quad\square$$

We record some explicit versions of the resolution $\mathbb{X}$ of Theorem 7.6. Keep in mind that if $p+p'=\boldsymbol{\alpha}-1$, then [16, Theorem 2.1] shows that the dimension of

$$\bar{B}_0(g+p')\cong\operatorname{Tor}^{\mathfrak{P}}_{p,p+e}(M_{-e},\boldsymbol{K})$$

can be computed from either of the split exact sequences:

$$\begin{array}{ll}0\to\operatorname{Tor}^{\mathfrak{P}}_{p,p+e}(M_{-e},\boldsymbol{K})\to\mathcal{N}(0,e,p)\to\cdots\to\mathcal{N}(p,p+e,0)\to 0\quad\text{or}\\ 0\to\mathcal{M}(g+p',p',0)\to\cdots\to\mathcal{M}(g,0,p')\to B_0(g+p')\to\bar{B}_0(g+p')\to 0.\end{array}\tag{7.9}$$

If $e=2$, then the Eagon-Northcott and Buchsbaum-Rim complexes (see [9, Theorem A2.10] or [16, Section 4]) give the resolution of M_ℓ for $-1\le\ell$. In particular, if $-1\le\ell$, then

$$\dim_{\boldsymbol{K}}\operatorname{Tor}^{\mathfrak{P}}_{p,q}(M_\ell,\boldsymbol{K})=\begin{cases}\dim S_{\ell-p}E_0\otimes\bigwedge^pG_0 & \text{if } q=p\text{ and }p\le\ell\\ \dim D_{p-\ell-1}E_0^*\otimes\bigwedge^{p+1}G_0 & \text{if }q=p+1\text{ and }\ell+1\le p\\ 0 & \text{otherwise.}\end{cases}$$

Example 7.10. To economize space, we write

$$T(p,q,\ell) \text{ for } \mathcal{P}'\otimes_{\boldsymbol{K}}\left[\operatorname{Tor}^{\mathfrak{P}}_{p,q}(M_\ell,\boldsymbol{K})\otimes_{\boldsymbol{K}}\bigwedge^{\ell+e}F_0^*\right].$$

If $e=g=2$, then $\mathbb{X}$ is

module	$\mathbb{X}_i$	twist	rank	module	$\mathbb{X}_i$	twist	rank
$\mathcal{P}'\otimes_{\boldsymbol{K}}\bar{B}_0(0)$	0	$[0,0]$	1	$T(2,4,-2)$	5	$[-2,-6]$	1
$\mathcal{P}'\otimes_{\boldsymbol{K}}\bar{B}_0(1)$	1	$[-1,-1]$	4	$T(1,3,-2)$	4	$[-1,-5]$	4
$T(0,0,0)$	1	$[0,-2]$	6	$T(1,2,0)$	4	$[-2,-4]$	6
$\mathcal{P}'\otimes_{\boldsymbol{K}}\bar{B}_0(2)$	2	$[-2,-2]$	3	$T(0,2,-2)$	3	$[0,-4]$	3
$T(0,1,-1)$	2	$[0,-3]$	8	$T(1,1,1)$	3	$[-2,-3]$	8
$T(0,0,1)$	2	$[-1,-2]$	8	$T(1,2,-1)$	3	$[-1,-4]$	8.

In other words, $\mathbb{X}$ is

$$0\longrightarrow \mathcal{P}'[-2,-6]\longrightarrow \begin{matrix}\mathcal{P}'[-2,-4]^6\\ \oplus\\ \mathcal{P}'[-1,-5]^4\end{matrix}\longrightarrow \begin{matrix}\mathcal{P}'[-2,-3]^8\\ \oplus\\ \mathcal{P}'[-1,-4]^8\\ \oplus\\ \mathcal{P}'[0,-4]^3\end{matrix}\longrightarrow \begin{matrix}\mathcal{P}'[-2,-2]^3\\ \oplus\\ \mathcal{P}'[-1,-2]^8\\ \oplus\\ \mathcal{P}'[0,-3]^8\end{matrix}\longrightarrow \begin{matrix}\mathcal{P}'[-1,-1]^4\\ \oplus\\ \mathcal{P}'[0,-2]^6\end{matrix}\longrightarrow \mathcal{P}'.$$

Example. Assume that $3\le e$ and $3\le g$. The beginning of $\mathbb{X}$ is

module	$\mathbb{X}_i$	twist	rank
$\mathcal{P}'\otimes_{\boldsymbol{K}}\bar{B}_0(0)$	0	$[0,0]$	1
$\mathcal{P}'\otimes_{\boldsymbol{K}}\bar{B}_0(1)$	1	$[-1,-1]$	eg
$T(0,0,0)$	1	$[0,-g]$	$\binom{f}{e}$
$\mathcal{P}'\otimes_{\boldsymbol{K}}\bar{B}_0(2)$	2	$[-2,-2]$	$\binom{eg}{2}$
$T(0,0,1)$	2	$[-1,-g]$	$e\binom{f}{e+1}$
$T(0,1,-1)$	2	$[0,-(g+1)]$	$g\binom{f}{e-1}$
$\mathcal{P}'\otimes_{\boldsymbol{K}}\bar{B}_0(3)$	3	$[-3,-3]$	see 7.9

$T(0,0,2)$	3	$[-2,-g]$	$\binom{e+1}{2}\binom{f}{e+2}$
$T(1,1,1)$	3	$[-2,-(g+1)]$	$\lvert\mathbb{M}(2,1)\rvert\binom{f}{e+1}$
$T(0,2,-2)$	3	$[0,-(g+2)]$	$\binom{f}{e-2}\binom{g+1}{2}$
$T(1,2,-1)$	3	$[-1,-(g+2)]$	$\lvert\mathbb{M}(1,2)\rvert\binom{f}{e-1}$.

The homology of the complex $\mathbb{M}(1,2)$ (see 7.3) is concentrated in one position and is equal to $\operatorname{Tor}^{\mathfrak{P}}_{1,2}(M_{-1},\boldsymbol{K})$; so, the dimension of $\operatorname{Tor}^{\mathfrak{P}}_{1,2}(M_{-1},\boldsymbol{K})$ is equal to the absolute value of the Euler characteristic of $\mathbb{M}(1,2)$.

Use (7.8) to calculate the contribution of the bottom summand to $\mathbb{X}_{eg+1}$:

$$\operatorname{Tor}^{\mathfrak{P}}_{eg-e,eg}(M_{-e},\boldsymbol{K}) = \mathrm{H}_{\mathcal{N}}(0,e,eg-e) = \boldsymbol{K}.$$

The largest index i for which the middle summand contributes to $\mathbb{X}_i$ is $i = eg$, and the contribution is $\bigwedge^g F^*[-e(g-1),-eg]$ because when $p = \boldsymbol{\alpha}$, $q = (e-1)g$, and $\ell = g-e$, then

$$\operatorname{Tor}^{\mathfrak{P}}_{p,q}(M_\ell,\boldsymbol{K}) = \mathrm{H}_{\mathcal{N}}(g-1,e-1,\boldsymbol{\alpha}) = \mathrm{H}_{\mathcal{M}}(0,0,0) = \boldsymbol{K}.$$

The end of $\mathbb{X}$ is

module	$\mathbb{X}_i$	twist	rank
$T(eg-e,eg,-e)$	$eg+1$	$[-(eg-e),-(eg+g)]$	1
$T(eg-e-1,eg-1,-e)$	eg	$[-(eg-e-1),-(eg+g-1)]$	eg
$T(\boldsymbol{\alpha},(e-1)g,g-e)$	eg	$[-(eg-e),-eg]$	$\binom{f}{e}$
$T(eg-e-2,eg-2,-e)$	$eg-1$	$[-(eg-e-2),-(eg+g-2)]$	$\binom{eg}{2}$
$T(\boldsymbol{\alpha},(e-1)g,g-e-1)$	$eg-1$	$[-(eg-e-1),-eg]$	$e\binom{f}{e+1}$
$T(\boldsymbol{\alpha},eg-g-1,g-e+1)$	$eg-1$	$[-(eg-e),-(eg-1)]$	$g\binom{f}{e-1}$
$T(eg-e-3,eg-3,-e)$	$eg-2$	$[-(eg-e-3),-(eg+g-3)]$	$\dim \bar{B}_0(3)$

$T(\boldsymbol{\alpha},(e-1)g,g-e-2)$	$eg-2$	$[-(eg-e-2),-eg]$	$\binom{e+1}{2}\binom{f}{e+2}$
$T(\boldsymbol{\alpha}-1,eg-g-1,g-e-1)$	$eg-2$	$[-(eg-e-2),-(eg-1)]$	$\lvert\mathbb{M}(2,1)\rvert\binom{f}{e+1}$
$T(\boldsymbol{\alpha},eg-g-2,g-e+2)$	$eg-2$	$[-(eg-e),-(eg-2)]$	$\binom{f}{e-2}\binom{g+1}{2}$
$T(\boldsymbol{\alpha}-1,eg-g-2,g-e+1)$	$eg-2$	$[-(eg-e-1),-(eg-2)]$	$\lvert\mathbb{M}(1,2)\rvert\binom{f}{e-1}$.

Example. Let $e=3$ and $g=4$. The module $T(3,4,-1)$ contributes the summand $\mathcal{P}'[-3,-8]^{210}$ to $\mathbb{X}_5$ and the module $T(2,4,-1)$ contributes the summand $\mathcal{P}'[-3,-8]^{420}$ to $\mathbb{X}_6$. By duality, $T(3,4,2)$ contributes the summand $\mathcal{P}'[-6,-8]^{210}$ to $\mathbb{X}_8$ and the module $T(4,4,2)$ contributes the summand $\mathcal{P}'[-6,-8]^{420}$ to $\mathbb{X}_7$. These summands can not be predicted if one only knows the Hilbert function of $\mathcal{R}'$. In this particular example, every other summand of the $\mathbb{X}$ can be correctly predicted from knowledge of the Hilbert function of $\mathcal{R}'$, together with the assumption that the minimal resolution of $\mathcal{R}'$ is as simple as possible.

Example. If $\boldsymbol{K}$ has characteristic zero, then the $\mathfrak{P}$-resolution of each module M_ℓ is known; and therefore, all of the modules in $\mathbb{X}$ are known in terms of Schur modules; see [19]. The paper [19] was inspired by the present paper; however, the proof is completely different. It uses the geometric method of finding syzygies and is valid only in characteristic zero. The resolution of Example 7.10 may also be found in [19].

Example. If e and g are both at least 5, then Hashimoto [10] proved that the dimension of $\operatorname{Tor}^{\mathfrak{P}}_{3,5}(M_0,\boldsymbol{K})$ depends on the characteristic of $\boldsymbol{K}$; and therefore, the graded betti number $\beta_8(5,g+5)$ in $\mathbb{X}$ depends on the characteristic of $\boldsymbol{K}$.

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References

1. L. Avramov, A. Kustin, and M. Miller, *Poincaré series of modules over local rings of small embedding codepth or small linking number*, J. Alg. **118** (1988), 162–204.
2. W. Bruns, *Divisors on varieties of complexes*, Math. Ann. **264** (1983), 53–71.
3. W. Bruns, *The existence of generic free resolutions and related objects*, Math. Scand. **55** (1984), 33–46.
4. W. Bruns and A. Guerrieri, *The Dedekind-Mertens formula and determinantal rings*, Proc. Amer. Math. Soc. **127** (1999), 657–663.
5. W. Bruns and U. Vetter, *Determinantal rings*, Lecture Notes in Mathematics **1327**, Springer Verlag, Berlin Heidelberg New York, 1988.

6. D. Buchsbaum and D. Eisenbud, *Some structure theorems for finite free resolutions*, Advances Math. **12** (1974), 84–139.
7. D. Buchsbaum and D. Eisenbud, *Generic free resolutions and a family of generically perfect ideals*, Advances Math. **18** (1975), 245–301.
8. D. Buchsbaum and D. Eisenbud, *Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension 3*, Amer. J. Math. **99** (1977), 447–485.
9. D. Eisenbud, *Commutative Algebra with a view toward Algebraic Geometry*, Graduate Texts in Mathematics **150**, Springer Verlag, Berlin Heidelberg New York, 1995.
10. M. Hashimoto, *Determinantal ideals without minimal free resolutions*, Nagoya Math. J. **118** (1990), 203–216.
11. R. Heitmann, *A counterexample to the rigidity conjecture for rings*, Bull. Amer. Math. Soc. (N.S.) **29** (1993), 94–97.
12. M. Hochster, *Topics in the homological theory of modules over commutative rings*, CBMS Regional Conf. Ser. in Math., no. 24, Amer. Math. Soc., Providence, RI, 1975.
13. C. Huneke, *The arithmetic perfection of Buchsbaum-Eisenbud varieties and generic modules of projective dimension two*, Trans. Amer. Math. Soc. **265** (1981), 211–233.
14. A. Kustin, *Ideals associated to two sequences and a matrix*, Comm. in Alg. **23** (1995), 1047–1083.
15. A. Kustin, *Complexes associated to two vectors and a rectangular matrix*, Mem. Amer. Math. Soc. **147** (2000), 1–81.
16. A. Kustin, *The cohomology of the Koszul complexes associated to the tensor product of two free modules*, Comm. in Algebra **33** (2005), 467–495.
17. A. Kustin and M. Miller, *Constructing big Gorenstein ideals from small ones*, J. Alg. **85** (1983), 303–322.
18. A. Kustin and M. Miller, *Multiplicative structure on resolutions of algebras defined by Herzog ideals*, J. London Math. Soc. (2) **28** (1983), 247–260.
19. A. Kustin and J. Weyman, *On the minimal free resolution of the universal ring for resolutions of length two*, preprint (ArXiv: math.AC/0508439).
20. P. Pragacz and J. Weyman, *On the generic free resolutions*, J. Alg. **128** (1990), 1–44.
21. A. Tchernev, *Universal complexes and the generic structure of free resolutions*, Mich. Math. J. **49** (2001), 65–96.

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