The Setup.

\( k \) is a field (For some applications \( k \) has to be infinite.)

\[
R = k[x_1, \ldots, x_n] \text{ is a polynomial ring}
\]

\( M \) is a finitely generated graded \( R \)-module

For example, \( M \) could be an ideal \( I \) of \( R \) which is generated by homogeneous polynomials or \( M \) could be a quotient ring \( R/I \) where again \( I \) is an ideal generated by homogeneous polynomials.
If one wants to “understand” $M$, one might want a minimal generating set for $M$. This would be a set of homogeneous elements $m_1, \ldots, m_{\beta_0}$ in $M$ so that every element in $M$ can be written “in terms” of $m_1, \ldots, m_{\beta_0}$ (and none of the $m_i$ can be omitted).

In particular, every element in $M$ has the form $\sum_{i=1}^{\beta_0} r_im_i$ for some $r_i$ in $R$.

As soon as one has a minimal generating set for $M$, the next natural question is “How can I tell when two elements of $M$ are the same?” That is, one wants to know the set

$$\left\{ \begin{bmatrix} r_1 \\ \vdots \\ r_{\beta_0} \end{bmatrix} \in R^{\beta_0} \mid \sum_{i=1}^{\beta_0} r_im_i = 0 \right\}.$$
The set
\[
\left\{ \begin{bmatrix} r_1 \\ \vdots \\ r_{\beta_0} \end{bmatrix} \in R^{\beta_0} \mid \sum_{i=1}^{\beta_0} r_i m_i = 0 \right\}
\]
is another finitely generated graded $R$-module, called the first syzygy module $\text{Syz}_1(M)$ of $M$.

Repeat the above process.

If one wants to “understand” $\text{Syz}_1(M)$, one might want a minimal generating set $X_1, \ldots, X_{\beta_1}$ for $\text{Syz}_1(M)$ and then one would want to know the relations on $X_1, \ldots, X_{\beta_1}$. The set of relations
\[
\left\{ \begin{bmatrix} r_1 \\ \vdots \\ r_{\beta_1} \end{bmatrix} \in R^{\beta_1} \mid \sum_{i=1}^{\beta_1} r_i X_i = 0 \right\}
\]
The set of relations

\[
\begin{cases}
  \begin{bmatrix}
    r_1 \\
    \vdots \\
    r_{\beta_1}
  \end{bmatrix} \in R^{\beta_1} & \sum_{i=1}^{\beta_1} r_i X_i = 0 \\
\end{cases}
\]

on \( X_1, \ldots, X_{\beta_1} \) is called the second syzygy module \( \text{Syz}_2(M) \) of \( M \).

One continues in this manner to find \( \text{Syz}_i(M) \) for all \( i \).

The syzygy modules of \( M \) are uniquely determined up to isomorphism.

The Hilbert syzygy theorem guarantees that \( \text{Syz}_i(M) = 0 \) for \( n + 1 \leq i \).

The numbers \( \{\beta_i\} \) are called the Betti numbers of \( M \).
The collection of free modules and induced maps

\[ \ldots \to R^{\beta_2} \to R^{\beta_1} \left[ X_1 \ldots X_{\beta_1} \right] \to R^{\beta_0} \]

is called the **minimal free resolution** of \( M \).

This might be a good time to say that in Astronomy “syzygy” refers to three celestial bodies in a straight line.

The word has Greek origin; it means “yoke”.

**Example** Let \( R = k[x,y] \) and \( M = R/(x^3, x^2y, y^2) \). Find the minimal homogeneous resolution of \( M \) by free \( R \)-modules.

It is clear that \( M \) is generated by the class \( \bar{1} \) of 1 in \( M \).

It is also clear that the multiples of \( \bar{1} \) that are zero in \( M \) are \( \theta \cdot \bar{1} \), where \( \theta \) is in the ideal \((x^3, x^2y, y^2)\) and that the listed generators are a minimal generating set of this ideal.

So, the ideal \((x^3, x^2y, y^2)\) is the first syzygy module of \( M \).
To find the second syzygy module of $M$, we look for

$$\begin{align*}
\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} \quad & r_1(x^3) + r_2(x^2 y) + r_3(y^2) = 0 \in R
\end{align*}$$

We notice that

$$\begin{align*}
y(x^3) & -x(x^2 y) = 0 \\
y(x^2 y) & -x^2(y) = 0
\end{align*}$$

One easily shows that

$$\begin{bmatrix} y \\ -x \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ y \\ -x^2 \end{bmatrix}$$

generate the entire second syzygy module of $M$. 
Indeed, if \( r_1(x^3) + r_2(x^2y) + r_3(y^2) = 0 \), then

\[
x^2(r_1x + r_2(x^2y)) + r_3(y^2) = 0
\]

in the UFD \( R \). So, \( x^2 \) divides \( r_3 \). Now one is left with

\( r_1(x^3) + r_2(x^2y) = 0 \). Again, one uses the fact that \( R \) is a UFD to show that

\[
\begin{bmatrix}
  r_1 \\
  r_2 \\
  0
\end{bmatrix}
\]

is a multiple of

\[
\begin{bmatrix}
  y \\
  -x \\
  0
\end{bmatrix}
\]

It is obvious that there are no non-zero relations on

\[
\begin{bmatrix}
  y \\
  -x \\
  0
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
  0 \\
  y \\
  -x^2
\end{bmatrix}
\].
We conclude that

\[
0 \rightarrow R(-4)^2 \xrightarrow{\begin{bmatrix} y & 0 \\ -x & y \\ 0 & -x^2 \end{bmatrix}} R(-3)^2 \oplus \begin{bmatrix} x^3 \\ x^2y \\ y^2 \end{bmatrix} \rightarrow R(-2)^1
\]

is the minimal homogeneous resolution of $M$. The numbers $(-2)$, $(-3)$, and $(-4)$ are keeping track of the degrees.
We can easily count $\dim_k M_i$ for each $i$

\[
\begin{align*}
\dim_k M_0 &= \dim_k R_0 = 1 \\
\dim_k M_1 &= \dim_k R_1 = 2 \\
\dim_k M_2 &= \dim_k R_2 - 1 \dim_k R_0 = 3 - 1 = 2 \\
\dim_k M_3 &= \dim_k R_3 - 1 \dim_k R_1 - 2 \dim_k R_0 = 4 - 2 - 2 = 0 \\
\dim_k M_i &= \dim_k R_i - 1 \dim_k R_{i-2} - 2 \dim_k R_{i-3} + 2 \dim_k R_{i-4} \\
&= (i + 1) - (i - 1) - 2(i - 2) + 2(i - 3) = 0 & \text{for } 4 \leq i
\end{align*}
\]
Of course, these counts are trivial for this $M$. As a vector space $M$ has basis

$$1, \quad x, y \quad x^2, xy.$$ 

But

1. The idea that the function $i \mapsto \dim_k M_i$ may be calculated from the minimal homogeneous resolution holds all the time. This function is called the **Hilbert function of** $M$, one could denote it $\text{HF}_M(i) = \dim_k M_i$.

2. The idea that there is a polynomial $\text{HP}_M(i)$ that gives the Hilbert function of $M$ for all large $i$ holds all the time. So, $\text{HP}_M(i)$ is a polynomial in $i$ and $\text{HP}_M(i) = \text{HF}_M(i)$ for all large $i$. The polynomial $\text{HP}_M(i)$ is called the **Hilbert polynomial** of $M$. The degree of $\text{HP}_M$ has geometric meaning:
The degree of $\text{HP}_M$ has geometric meaning: $\deg \text{HP}_M(i) = \text{Kdim}(M) - 1$. I have used $\text{Kdim}$ to represent Krull dimension. The Krull dimension of (the coordinate ring of) a finite set of points (in affine space) is zero; the Krull dimension of (the coordinate ring of) a curve (in affine space) is 1; the Krull dimension of (the coordinate ring of) a surface (in affine space) is two; the Krull dimension of $k[x_1, \ldots, x_n]$ is $n$. The leading coefficient of $\text{HP}_M(i)$ also has geometric meaning. This leading coefficient is $\frac{e(M)}{(\text{Kdim}(M) - 1)!}$. Commutative algebraists call $e(M)$ the multiplicity of $M$; Geometers call $e(M)$ the degree of $M$. 
3. The idea that the “generating function” $HS_M(t) = \sum_i (\dim_k M_i) t^i$
(called the **Hilbert series** of $M$) is equal to

   a polynomial that may be read from the resolution $\cdot HS_R(t)$

   \[ = \frac{a \text{ polynomial that may be read from the resolution}}{(1 - t)^n} \]

holds all of the time.
Example. Assume that $k$ is infinite. Consider the morphism $\Gamma : \mathbb{P}^1 \rightarrow \mathbb{P}^3$ which is given by $\Gamma([s:t]) = [s^3 : s^2t : st^2 : t^3]$. Let $C$ be the image of $\Gamma$. Let $R = k[x_1, x_2, x_3, x_4]$ be the homogeneous coordinate ring for $\mathbb{P}^3$.

(a) Find $I(C) = \{ f \in R | f \text{ vanishes on } C \}$.

(b) Find the minimal homogeneous resolution of $S = R/I(C)$. (The ring $S$ is called the homogeneous coordinate ring of $C$.) Observe that $C$ is a curve, so the coordinate ring of some dehomogenization of $C$ has Krull dimension 1 and $S$ has Krull dimension 2.

(c) Compute the multiplicity of $S$.

(d) Does it make sense to say that the degree of $C$ is equal to the answer to (c)?
\[ C = \{ [s^3 : s^2 t : st^2 : t^3] \in \mathbb{P}^3 \mid [s : t] \in \mathbb{P}^1 \} \]

(a) We see that \( x_1 x_3 - x_2^2, x_1 x_4 - x_2 x_3, \) and \( x_2 x_4 - x_3^2 \) all vanish on \( C. \)

We see that every element of \( R \) has the form

\[
f_1(x_1, x_4) + x_2 f_2(x_1, x_4) + x_3 f_3(x_1, x_4)
\]

\[+\] an element of \((x_1 x_3 - x_2^2, x_1 x_4 - x_2 x_3, x_2 x_4 - x_3^2)\)

No terms in

\[
f_1(s^3, t^3) + s^2 t f_2(s^3, t^3) + st^2 f_3(s^3, t^3) \quad (1)
\]

can cancel; so (1) vanishes on \( C \) only if

\( f_1(x_1, x_4) + x_2 f_2(x_1, x_4) + x_3 f_3(x_1, x_4) \) is the zero polynomial.

Conclude \( I(C) = (x_1 x_3 - x_2^2, x_1 x_4 - x_2 x_3, x_2 x_4 - x_3^2). \)
\[ I(C) = (x_1 x_3 - x_2^2, x_1 x_4 - x_2 x_3, x_2 x_4 - x_3^2) \]

(b)

\[
\begin{pmatrix}
    x_1 & x_2 \\
    x_2 & x_3 \\
    x_3 & x_4 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
    x_2 & x_3 \\
    x_3 & x_4 \\
\end{pmatrix} \rightarrow R(-2)^3
\]

\[
\begin{pmatrix}
    x_1 & x_2 \\
    x_3 & x_4 \\
\end{pmatrix} \rightarrow R
\]

(c)

\[
HS_S(t) = \frac{1 - 3t^2 + 2t^3}{(1 - t)^4} = \frac{1 + 2t}{(1 - t)^2}
\]

\[
e_R(S) = (1 + 2t)|_1 = 3.
\]
(d) Yes, \(C\), which is parameterized by
\[
\Gamma([s : t]) = [s^3 : s^2 t : st^2 : t^3],
\]
appears to be a curve of degree 3.

**Example.** This is the first example in Eisenbud’s book “The geometry of syzygies”.

It shows that syzygies encode significant information about geometry. Let \(X\) be a set of 7 points in projective 3-space \(\mathbb{P}^3\) over \(k\). Assume no more than 2 points of \(X\) are on any line and no more than 3 points of \(X\) are on any plane.

Let \(I(X)\) be the set of polynomials in \(R = k[x_1, x_2, x_3, x_4]\) that vanish on \(X\) and \(S\) be the homogeneous coordinate ring of \(X\), in other words, let \(S = R/I(X)\).

View \(S\) as an \(R\)-module.
\[ X \subseteq \mathbb{P}^3, \text{ 7 points in general position, } S = R/I(X), \quad R = k[x_1, \ldots, x_4] \]

Eisenbud says there are exactly two distinct configurations for graded Betti numbers of the homogeneous coordinate ring \( S \):

\[
0 \longrightarrow R(-5)^3 \longrightarrow R(-4)^6 \longrightarrow \bigoplus \longrightarrow R
\]

\[
R(-2)^3
\]

\[
R(-3)
\]

\[
0 \longrightarrow R(-5)^3 \longrightarrow \bigoplus \longrightarrow \bigoplus \longrightarrow R
\]

\[
R(-3)^2 \quad R(-2)^3
\]

\[
R(-4)^6 \quad R(-3)^3
\]

In the first case the points do not lie on any curve of degree 3. In the second case, the ideal \( J \) generated by the three quadratic generators of \( I(X) \) is the ideal of the unique curve of “degree 3”
which contains $X$ and this curve is irreducible. I think that Eisenbud claims that after a change of variables, the unique curve of “degree 3” is the $C$ from the earlier example.

**Example.** This example shows that the form of the syzygies and not just the graded Betti numbers affects the geometry. Consider three homogeneous polynomials $g_1, g_2, g_3$ of the same degree in $k[x,y]$. Assume that the only polynomials that divide all three $g$’s are the constants. The morphism

$$[g_1 : g_2 : g_3] : \mathbb{P}^1 \rightarrow \mathbb{P}^2$$

$$[a : b] \mapsto [g_1(a,b) : g_2(a,b) : g_3(a,b)]$$

defines a rational curve $C$ in the projective plane. Information about the singularities of $C$ may be read from the syzygies of $[g_1, g_2, g_3]$. Let $\varphi$ denote the syzygy matrix; as in the other examples, the $g$’s are the signed maximal order minors of $\varphi$. .
The configuration of all singularities that can appear on, or infinitely near, a rational plane quartic are completely determined by two classical formulas:

\[ g = \binom{d - 1}{2} - \sum_q \binom{m_q}{2} \quad \text{and} \quad \sum_{q'} m_{q'} \leq m_p, \]

The formula on the left was known by Max Noether. It gives the genus \( g \) of the irreducible plane curve \( C \) of degree \( d \), where \( q \) varies over all singularities on, and infinitely near, \( C \), and \( m_q \) is the multiplicity at \( q \). The formula on the right holds for any point \( p \) on any curve \( C \); the points \( q' \) vary over all of the points in the first neighborhood of \( p \). The above formulas permit 9 possible
The thirteen possible ways to configure the above singularities on a rational plane curve of degree 4:

<table>
<thead>
<tr>
<th>Classical name</th>
<th>modern name</th>
<th>multiplicity sequence</th>
</tr>
</thead>
<tbody>
<tr>
<td>Node</td>
<td>$A_1$</td>
<td>$(2 : 1, 1)$</td>
</tr>
<tr>
<td>Cusp</td>
<td>$A_2$</td>
<td>$(2 : 1)$</td>
</tr>
<tr>
<td>Tacnode</td>
<td>$A_3$</td>
<td>$(2 : 2 : 1, 1)$</td>
</tr>
<tr>
<td>Ramphoid Cusp</td>
<td>$A_4$</td>
<td>$(2 : 2 : 1)$</td>
</tr>
<tr>
<td>Oscnode</td>
<td>$A_5$</td>
<td>$(2 : 2 : 2 : 1, 1)$</td>
</tr>
<tr>
<td>$A_6$-Cusp</td>
<td>$A_6$</td>
<td>$(2 : 2 : 2 : 1)$</td>
</tr>
<tr>
<td>Ordinary Triple Point</td>
<td>$D_4$</td>
<td>$(3 : 1, 1, 1)$</td>
</tr>
<tr>
<td>Tacnode Cusp</td>
<td>$D_5$</td>
<td>$(3 : 1, 1)$</td>
</tr>
<tr>
<td>Multiplicity 3 Cusp</td>
<td>$E_6$</td>
<td>$(3 : 1)$</td>
</tr>
</tbody>
</table>
rational plane quartic are given by

\[(3 : 1, 1, 1) \quad (2 : 2 : 2 : 1, 1) \quad (2 : 2 : 1, 1), (2 : 1, 1) \quad (2 : 1, 1)^3\]

\[(3 : 1, 1) \quad (2 : 2 : 2 : 1) \quad (2 : 2 : 1, 1), (2 : 1) \quad (2 : 1, 1)^2, (2 : 1)\]

\[(3 : 1) \quad (2 : 2 : 1), (2 : 1, 1) \quad (2 : 1, 1), (2 : 1)^2\]

\[(2 : 1)^3 \quad (2 : 2 : 1), (2 : 1).\]

If the singularity configuration of \( C \) is given by is equal to

\[(2 : 2 : 2 : 1, 1) \quad \text{or} \quad (2 : 2 : 2 : 1)\]

then

\[
\phi = \begin{bmatrix}
Q_1 & Q_2 \\
Q_3 & Q_1 \\
0 & Q_3
\end{bmatrix}.
\]
If the singularity configuration of $C$ is given by is equal to

$$(2 : 2 : 2 : 1, 1) \quad \text{or} \quad (2 : 2 : 2 : 1)$$

then

$$\phi = \begin{bmatrix} Q_1 & Q_2 \\ Q_3 & Q_1 \\ 0 & Q_3 \end{bmatrix}.$$ 

(a) If the singularity configuration is $(2 : 2 : 2 : 1, 1)$, then $Q_3$ is the product of 2 non-associate linear forms in $B$. The only singularity on $C$ is $[0 : 0 : 1]$ and this singularity is an oscnode.

(b) If the singularity configuration is $= (2 : 2 : 2 : 1)$, then $Q_3$ is a perfect square. The only singularity on $C$ is $[0 : 0 : 1]$ and this singularity is an $A_6$-cusp.
Let $C$ be parameterized by the m.o.m. of $\varphi = \begin{bmatrix} -(x-y)^2 & -xy \\ x^2 & -(x-y)^2 \\ 0 & x^2 \end{bmatrix}$. This singularity has one branch only after 3 blowups. It is an “$A_6$-cusp”.

The picture looks like a “thought caption”.

Let $C$ be parameterized by the m.o.m. of $\varphi = \begin{bmatrix} x^2+xy+y^2 & y^2-x^2 \\ x(x+2y) & x^2+xy+y^2 \\ 0 & x(x+2y) \end{bmatrix}$. This singularity has two branches after 3 blowups. It is an “oscnode”. The picture looks like a “bowtie”.
with(algcurves);
[AbelMap, Siegel, Weierstrassform, algfun_series_sol, differentials, genus, homogeneous, homology, implicitize, integral_basis, is_hyperelliptic, j_invariant, monodromy, parametrization, periodmatrix, plot_knot, plot_real_curve, puiseux, singularities]

plot_real_curve((x^2 + y^2)^2 - x*y - x^3 - 2*x^2*y + x^2, x, y);
with(algcurves);

[AbelMap, Siegel, Weierstrassform, algfun_series_sol, differentials, genus,
  homogeneous, homology, implicitize, integral_basis, is_hyperelliptic,
  j_invariant, monodromy, parametrization, periodmatrix, plot_knot,
  plot_real_curve, puiseux, singularities]

plot_real_curve(x^3*(x-1) + (y^2-x)^2, x, y);
So much for the questions “What is a syzygy?” and “Why do I care?”

Now it is time to answer, “So how does one find syzygies?”

I propose four methods.

1. One can build complicated resolutions from simpler resolutions. If $M = M_1/M_2$ and one knows how that $\mathcal{B}$ resolve $M_1$ and $\mathcal{T}$ resolves $M_2$, then $M$ is resolved by the mapping cone of

\[
\begin{array}{c}
\cdots \\
\downarrow c_1 \\
\downarrow c_1 \\
\downarrow c_0 \\
\cdots \\
\end{array}
\begin{array}{c}
\downarrow b_2 \\
\downarrow b_1 \\
\downarrow c_0 \\
\cdots \\
\end{array}
\begin{array}{c}
\longrightarrow T_2 \\
\longrightarrow t_2 \\
\longrightarrow T_1 \\
\longrightarrow t_1 \\
\longrightarrow T_0 \\
\longrightarrow b_1 \\
\longrightarrow B_0.
\end{array}
\]
2. One can use methods from Algebraic Topology to find syzygies. The starting point for this idea is the following Theorem of Hochster. If $I$ is a square-free monomial ideal in $R$, then $I$ corresponds to a simplicial complex $\Delta$ and the Betti numbers of $R/I$ may be computed from the reduced simplicial homology of $\Delta$.

3. A computer uses Gröbner bases to compute syzygies. One orders the monomials in $R$. Every polynomial now has a leading term. One picks a generating set

$$g_1, \ldots, g_N \quad (2)$$

for $I$ with the property that $\text{leading-term}(g_1), \ldots, \text{leading-term}(g_N)$ is also a generating set for the ideal $(\{\text{leading-term}(f) \mid f \in I\})$. The generating set (2) is called a Gröbner basis for $I$. It is not a minimal generating set for $I$, but it is wonderful for making calculations.
4. One can use the Geometric method for finding syzygies. Let $Y$ be a subvariety of $\mathbb{P}^{n-1}$. To employ the geometric method for finding the resolution of $R/I(Y)$ by free $R$-modules, one first resolves the singularity. Let $Z \to Y$ be a desingularization of $Y$. Now, $Z$ is smooth; so, $Z$ is defined by a regular sequence and the coordinate ring of $Z$ is resolved by the Koszul complex for this regular sequence. The Koszul complex resolution is the most completely understood and straightforward of all possible resolutions. To resolve the homogeneous coordinate ring of $Y$, one “need only” push the Koszul complex down to $Y$ along the desingularization.