THE RESOLUTION OF THE BRACKET POWERS OF THE MAXIMAL IDEAL IN A DIAGONAL HYPERSURFACE RING

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ABSTRACT. Let $\mathbf{k}$ be a field. For each pair of positive integers (n, N), we resolve $Q = R/(x^N, y^N, z^N)$ as a module over the ring $R = \boldsymbol{k}[x, y, z]/(x^n + y^n + z^n)$. Write N in the form $N = an + r$ for integers a and r, with r between 0 and $n - 1$. If n does not divide N and the characteristic of $\boldsymbol{k}$ is fixed, then the value of a determines whether Q has finite or infinite projective dimension. If Q has infinite projective dimension, then value of r, together with the parity of a, determines the periodic part of the infinite resolution. When Q has infinite projective dimension we give an explicit presentation for the module of first syzygies of Q. This presentation is quite complicated. We also give an explicit presentation the module of second syzygies for Q. This presentation is remarkably uncomplicated. We use linkage to find an explicit generating set for the grade three Gorenstein ideal $(x^N, y^N, z^N) : (x^n + y^n + z^n)$ in the polynomial ring $\boldsymbol{k}[x, y, z]$.

The question "Does Q have finite projective dimension?" is intimately connected to the question "Does $\boldsymbol{k}[X, Y, Z]/(X^a, Y^a, Z^a)$ have the Weak Lefschetz Property?". The second question is connected to the enumeration of plane partitions.

When the field $\boldsymbol{k}$ has positive characteristic, we investigate three questions about the Frobenius powers $F^t(Q)$ of Q. When does there exist a pair (n, N) so that Q has infinite projective dimension and $F(Q)$ has finite projective dimension? Is the tail of the resolution of the Frobenius power $F^t(Q)$ eventually a periodic function of t, (up to shift)? In particular, we exhibit a situation where the tail of the resolution of $F^t(Q)$, after shifting, is periodic as a function of t, with an arbitrarily large period. Can one use socle degrees to predict that the tail of the resolution of $F^t(Q)$ is a shift of the tail of the resolution of Q?

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1991 *Mathematics Subject Classification.* 13A35, 13D02, 13H10, 13C40, 13E10, 13D40.

Key words and phrases. Almost Complete Intersection, Enumeration of Plane Partitions, Frobenius Periodicity, Frobenius Power, Hilbert-Burch Matrix, Hilbert-Kunz Functions, Pfaffians, Socle Degrees, Syzygy, Syzygy Gap, Weak Lefschetz Property.

[1]Supported in part by the National Security Agency.

[2]Supported in part by the National Security Agency.

Section 0. Introduction.

Throughout this paper the diagonal hypersurface ring $R_{\boldsymbol{k},n}$ is the ring

$$R_{\boldsymbol{k},n} = \boldsymbol{k}[x,y,z]/(x^n + y^n + z^n), \tag{0.1}$$

where $\boldsymbol{k}$ is a field and n is a positive integer. For each positive integer N, we resolve the quotient ring

$$Q_{\boldsymbol{k},n,N} = R_{\boldsymbol{k},n}/(x^N, y^N, z^N), \tag{0.2}$$

as a module over $R_{\boldsymbol{k},n}$. For each real number α which is not of the form $b + \frac{1}{2}$ for some integer b, let $\{\alpha\}$ represent the integer which is closest to α. In particular, if b is an integer, then

$$\{\tfrac{b}{3}\} = \begin{cases} \frac{b}{3} & \text{if } b \equiv 0 \mod 3 \\ \frac{b-1}{3} & \text{if } b \equiv 1 \mod 3 \\ \frac{b+1}{3} & \text{if } b \equiv 2 \mod 3. \end{cases} \tag{0.3}$$

One of our main results is

Theorem 6.3. *Let $\boldsymbol{k}$ be a field of characteristic c and let n and N be positive integers. Then $\operatorname{pd}_{R_{\boldsymbol{k},n}} Q_{\boldsymbol{k},n,N}$ is finite if and only if at least one of the following conditions hold*:

(1) *n divides N, or*
(2) *$c = 2$ and $n \le N$, or*
(3) *$c = p$ is an odd prime and there exist an odd integer J and a power $q = p^e$ of p with $e \ge 1$ and $|Jq - \frac{N}{n}| < \{\frac{q}{3}\}$.*

In particular, if $c = 0$, then $\operatorname{pd}_{R_{\boldsymbol{k},n}} Q_{\boldsymbol{k},n,N}$ is finite if and only if n divides N.

We use pd to mean *projective dimension*. When our meaning is clear, we write R and Q in place of $R_{\boldsymbol{k},n}$ and $Q_{\boldsymbol{k},n,N}$, respectively. Write N in the form $N = \theta n + r$, for integers θ and r, with $0 \le r \le n-1$. If n does not divide N and the characteristic of $\boldsymbol{k}$ is fixed, then the value of θ determines whether $Q_{\boldsymbol{k},n,N}$ has finite or infinite projective dimension. If $Q_{\boldsymbol{k},n,N}$ has infinite projective dimension, then value of r, together with the parity of θ, determines the periodic part of the infinite resolution.

There are three steps in the proof of Theorem 6.3. In Definitions 1.1 and 1.10 we define sets of non-negative integers S_c and T_c which depend on the characteristic c of $\boldsymbol{k}$. (We use c to be the characteristic of the field $\boldsymbol{k}$; thus c is either 0 or a positive prime integer p.) The first step is Theorem 3.5, where we record an explicit infinite homogeneous minimal resolution of Q when θ is in S_c and $1 \le r$. The second step is Theorem 5.14 where we identify a set of integers T_p such that if $\theta \in T_p$, then $\operatorname{pd}_R Q < \infty$. The proof in the finite projective dimension case is less explicit than the proof in the infinite projective dimension case. In the finite projective dimension case we use a Theorem of Brenner and Kaid which identifies those integers a for which the Hilbert-Burch matrix HB_a for $[X^a, Y^a, (X+Y)^a]$ in $\boldsymbol{k}[X,Y]$ is unbalanced. (See 5.1 for the definitions.) The paper of Brenner and Kaid [1] ties this unbalanced behavior of the Hilbert-Burch matrix to the absence of the Weak Lefschetz Property (WLP) in the ring $\bar{\boldsymbol{k}}[X,Y,Z]/(X^a, Y^a, Z^a)$, where $\bar{\boldsymbol{k}}$ is the algebraic closure of $\boldsymbol{k}$. (We notice that Li and Zanello [9] have found "a surprising, and still combinatorially obscure, connection" between the monomial complete intersection ideals in three variables which satisfy the WLP, as a function of the characteristic of the base field, and the enumeration of plane partitions.) The proof of the Brenner-Kaid result uses a Theorem of Han [5]. An easier proof of Han's Theorem was given by Monsky [10]. The result in the present paper which connects unbalanced Hilbert-Burch matrices to $\operatorname{pd}_R Q < \infty$ is Theorem 5.6. The set T_p contains and is slightly larger than the set of indices a where HB_a is unbalanced.

Each set S_p is defined recursively as a union of intervals. Each set T_p is defined explicitly as a union of intervals. It is not immediately obvious that the union of S_p and T_p is the set of all non-negative integers. The proof of this fact, which is the third and final step in the proof of Theorem 6.3, is given in Theorem 6.1. The proof of Theorem 6.1 makes use of a base-p expansion of integers using unusual coefficients; see Notation 1.5.

Three ingredients are used in the proof of Theorem 3.5, which is the infinite projective dimension case. The linkage part of the argument, which is similar to the work of Buchsbaum-Eisenbud (see for example [3, Thm. 5.3]), is carried out in Section 2. In order to apply Section 2, one must find a matrix ψ and a unit u so that the quadratic equation that we have called (2.2) is satisfied. In Section 3 we propose a solution to the problem posed by (2.2). We prove that our proposed solution works over $\mathbb{Z}$ in Section 7. In Section 4 we prove that when $\theta \in S_p$, then the solution of (2.2) which works over $\mathbb{Z}$ can be made to work over $\boldsymbol{k}$.

The critical parts of the proof of Theorem 3.5 take place in the polynomial ring

$P = \boldsymbol{k}[x,y,z]$. In particular, we determine a generating set for the grade three Gorenstein ideal

$$(x^N, y^N, z^N) : (x^n + y^n + z^n) \tag{0.4}$$

in P. We prove (see Remark 2.13) that $Q_{\boldsymbol{k},n,N}$ has infinite projective dimension over $R_{\boldsymbol{k},n}$ if and only if the ideal of (0.4) of P has seven generators.

One consequence of our work (see Corollary 3.7) is that the module of second syzygies of the $R_{\boldsymbol{k},n}$-module $Q_{\boldsymbol{k},n,N}$ is remarkably uncomplicated. Each such second syzygy module is either free or is presented by a matrix of the form

$$\varphi_{r,s} = \begin{bmatrix} 0 & z^r & -y^r & x^s \\ -z^r & 0 & x^r & y^s \\ y^r & -x^r & 0 & z^s \\ -x^s & -y^s & -z^s & 0 \end{bmatrix},$$

where r and s are positive integers with $r+s=n$. This fact is somewhat surprizing because the presentation matrix for the module of first syzygies of $Q_{\boldsymbol{k},n,N}$ is very complicated. Its entries involve high degrees, many terms, and intricate coefficients given in terms of binomial coefficients. The module of first syzygies of $Q_{\boldsymbol{k},n,N}$ is studied in a very recent preprint of Brenner and Kaid [2]. They use the syzygy bundle $\operatorname{Syz}(x^p, y^p, z^p)$ on $\operatorname{Proj}(R_{\boldsymbol{k},n})$, where $\boldsymbol{k}$ is an algebraically closed field of characteristic p to create vector bundles $\mathcal{E}$ with $\mathcal{E}$ isomorphic to $F^*(\mathcal{E})$, where $F\colon R_{\boldsymbol{k},n} \to R_{\boldsymbol{k},n}$ is the Frobenius functor.

Recall that if R is a ring of positive characteristic p, J is an ideal in R, and $q = p^e$, for some positive integer e, then the e^{th} Frobenius power of J is the ideal $J^{[q]}$ generated by all j^q with $j \in J$. Furthermore, the e^{th} Frobenius power of the $R_{\boldsymbol{k},n}$-module $Q_{\boldsymbol{k},n,N}$ is $Q_{\boldsymbol{k},n,qN}$. Three questions pertaining to Frobenius powers are investigated in Section 8. The first question is "When does the Cohen-Macaulay local ring $(R,\mathfrak{m})$ have an $\mathfrak{m}$-primary ideal J so that R/J has infinite projective dimension but the Frobenius power $R/J^{[p]}$ has finite projective dimension?" It is shown that if R is not F-injective, then such an ideal exists. Furthermore, when R is one of the rings $R_{\boldsymbol{k},n}$, then there exists an integer N so that $J = (x^N, y^N, z^N)$ has the above property if and only if R is not F-injective. The second questions is "Is the tail of the resolution of the Frobenius power $Q_{\boldsymbol{k},n,p^tN}$ (up to shift) eventually a periodic function of t?" The answer is yes. The third question is "Can one use socle degrees to predict that the tail of the resolution of $Q_{\boldsymbol{k},n,p^tN}$ is a shift of the tail of the resolution of $Q_{k,n,N}$?" The third question was answered in [8] in the case that both modules have finite projective dimension (hence the infinite tail of both resolutions is zero). It is shown in [7] how the socle degrees can be used to predict that the tail of the resolution of $Q_{\boldsymbol{k},n,p^tN}$ is a shift of the tail of the resolution of

$Q_{\boldsymbol{k},n,N}$, **as a graded module**. We show that the condition of [7], when applied to $Q_{\boldsymbol{k},n,N}$, actually produces an isomorphism of complexes **with differential**.

In Section 9 we resolve the module $\bar{Q}_{\boldsymbol{k},n,N} = \bar{R}_{\boldsymbol{k},n}/(x^N, y^N)$ over the two variable diagonal hypersurface ring $\bar{R} = \boldsymbol{k}[x,y]/(x^n + y^n)$ and we prove that the tail of the resolution of the Frobenius power $F^t(\bar{Q}_{\boldsymbol{k},n,N})$ is isomorphic to a shift of the tail of the resolution of $\bar{Q}_{\boldsymbol{k},n,N}$ as a graded module if and only if these objects are isomorphic as complexes if and only if the socle of $F^t(\bar{Q}_{\boldsymbol{k},n,N})$ is isomorphic to a shift of the socle of $\bar{Q}_{\boldsymbol{k},n,N}$. We conclude by exhibiting a situation where the tail of the resolution of $F^t(\bar{Q}_{\boldsymbol{k},n,N})$, after shifting, is periodic as a function of t, with an arbitrarily large period. The results of Section 9 were announced in [7].

Section 1. Terminology, notation, and elementary results.

In this section we gather the terminology notation, and elementary results which are used throughout the paper. We begin by defining the sets S_c and T_c which appear in the statement of Theorem 6.2 and in the proof of Theorem 6.3. **In this discussion c is the characteristic of a field so c is either zero or a positive prime integer p.**

Definition 1.1. For each field $\boldsymbol{k}$, we define a set of non-negative integers S_c, where c is the characteristic of $\boldsymbol{k}$. If c is a positive prime integer p, with $p \geq 3$, then we also define a second set of non-negative integers D_p.

(1) The set S_0 is the set of all non-negative integers.

(2) The set S_2 is $\{0\}$.

(3) The sets D_3 and S_3 are defined recursively.

(a) Define D_3 as follows. The number 0 is in D_3 and if $d \in D_3$, then $3d$ and $3d+2$ are also in D_3.

(b) Define S_3 as follows. The number 0 is in S_3 and if a is an even element of S_3, then $3a$, $3a+1$, $3a+4$, and $3a+5$ all are in S_3.

(4) Let $p \geq 5$ be prime. Define $\pi_p = \pi$ to be the largest integer with $\pi_p < \frac{p}{3}$. The sets D_p and S_p are defined recursively.

(a) Define D_p as follows. The closed interval of integers $[0, \pi] \subseteq D_p$; and if d is a non-negative element of D_p, then $[pd - \pi, pd + \pi] \subseteq D_p$.

(b) Define S_p as follows. The closed interval of integers $[0, 2\pi] \subseteq S_p$; and if θ is a non-negative even element of S_p, then $[p\theta - 2\pi - 1, p\theta + 2\pi] \subseteq S_p$.

Remarks 1.2.

(1) The sets D_p and S_p are related as follows:

$$\begin{cases} D_p = \{\frac{1}{2}\theta \mid \theta \text{ is an even element of } S_p\}, & \text{for } 3 \le p, \\ S_3 = \{2d \mid d \in D_3\} \cup \{2d+1 \mid d \in D_3\}, \\ S_p = \{2d \mid d \in D_p\} \cup \{2d-1 \mid d \in D_p\}, & \text{for } 5 \le p. \end{cases} \tag{1.3}$$

One could also define exactly one of the sets D_p or S_p in a recursive manner and then define the other set using (1.3).

(2) If a is an odd integer and $p \geq 3$ is a prime integer, then

$$\begin{cases} a \in S_p \iff a-1 \in S_p, & \text{if } p = 3, \text{ and} \\ a \in S_p \iff a+1 \in S_p, & \text{if } p \geq 5. \end{cases}$$

Example 1.4. We see that

$$\begin{array}{ll} & D_3 = \{0,2,6,8,18,20,24,26,54,56,60,62,72,74,78,80,\dots\}, \\ & S_3 = \{0,1,4,5,12,13,16,17,36,37,40,41,48,49,52,53,\dots\}, \\ \pi_5 = 1, & D_5 = [0,1] \cup [4,6] \cup [19,21] \cup [24,26] \cup \dots, \\ & S_5 = [0,2] \cup [7,12] \cup [37,42] \cup [47,52] \cup \dots, \\ \pi_7 = 2, & D_7 = [0,2] \cup [5,9] \cup [12,16] \cup [33,37] \cup \dots, \\ & S_7 = [0,4] \cup [9,18] \cup [23,32] \cup [65,74] \cup \dots, \\ \pi_{11} = 3, & D_{11} = [0,3] \cup [8,14] \cup [19,25] \cup [30,36] \cup \dots, \\ & S_{11} = [0,6] \cup [15,28] \cup [37,50] \cup [59,72] \cup \dots, \\ \pi_{13} = 4, & D_{13} = [0,4] \cup [9,17] \cup [22,30] \cup [35,43] \cup \dots, \text{ and} \\ & S_{13} = [0,8] \cup [17,34] \cup [43,60] \cup [69,86] \cup \dots\ . \end{array}$$

In Remarks 1.6 and 1.7 we offer an alternate, explicit, description of S_p for $p \geq 3$.

Notation 1.5. Let $p \geq 5$ be a prime number. We will write integers in base-p, using even digits of the form $-2k, \dots, -2, 0, 2, \dots, 2k$, where $2k = v$ is the largest even integer such that $2k < 2p/3$, and odd digits of the form $-u, \dots, -1, 1, \dots, u$, where u is the largest odd integer such that $u < p/3$. It is easy to see that every integer can be written uniquely in the form $a_0 + a_1p + \ldots + a_tp^t$, with $a_0, a_1, \dots, a_t$ digits as above.

Remark 1.6. Fix a prime number $p \geq 5$. Recall the set S_p of Definition 1.1. It is not difficult to see that if m is a non-negative even integer, then $m \in S_p$ if and only if the base-p expansion of m, in the sense of Notation 1.5, involves only even digits.

Remark 1.7. *If a is a non-negative integer, then $a \in S_3$ if and only if there are coefficients ϵ_i and ϵ in the set $\{0,1\}$ such that*

$$a = \epsilon + 4\sum_{i=0}^{r} \epsilon_i 3^i.$$

Proof. ($\Rightarrow$) This direction proceeds by induction. It is clear that 0 has the correct form. If the even element a has the correct form $a = 4\sum_{i=0}^{r} \epsilon_i 3^i$, then the integers

$$\begin{aligned}
&3a = 4\textstyle\sum_{i=0}^{r} \epsilon_i 3^{i+1},\\
&3a+1 = 1 + 4\textstyle\sum_{i=0}^{r} \epsilon_i 3^{i+1},\\
&3a+4 = 4(\textstyle\sum_{i=0}^{r} \epsilon_i 3^{i+1} + 1), \text{ and}\\
&3a+5 = 1 + 4(\textstyle\sum_{i=0}^{r} \epsilon_i 3^{i+1} + 1)
\end{aligned}$$

all also have the correct form.

($\Leftarrow$) The recursive definition of S_3 shows that the integers

$$4\epsilon_r, \quad 4(3\epsilon_r + \epsilon_{r-1}), \quad 4(3^2\epsilon_r + 3\epsilon_{r-1} + \epsilon_{r-2}), \quad \dots$$

are all in S_3. $\square$

The sets S_p are constructed to ensure that certain formulas about the divisibility of binomial coefficients by p hold for $\theta \in S_p$; see Theorem 1.9. We introduce the relevant ideas at this point.

Definition 1.8. If M is an integer and p a prime number, then we define $M_{\#p}$ as follows:

$$M_{\#p} = \sup\{k \mid \ p^k \text{ divides } M\}.$$

In particular, $0_{\#p} = \infty$.

Theorem 1.9. *Let $d \in D_p$ for some prime $p \geq 3$.*

(1) *If $p = 3$, then*
   - **(a)** $\binom{2d}{d}_{\#3} = \binom{3d}{d}_{\#3}$,
   - **(b)** $\binom{2d+1}{d}_{\#3} = \binom{3d+2}{d}_{\#3}$,
   - **(c)** $\left(\binom{a}{d}\binom{3d-a}{d}\right)_{\#3} \geq \binom{2d}{d}_{\#3}$,
   - **(d)** $\left(\binom{a}{d}\binom{3d-1-a}{d}\right)_{\#3} \geq \binom{2d}{d}_{\#3}$, *and*
   - **(e)** $\left(\binom{a}{d}\binom{3d+1-a}{d+1}\right)_{\#3} \geq \binom{2d+1}{d}_{\#3}$,

for all integers a with $0 \le a \le 2d$.

(2) *If $p \ge 5$, then*

(a) $\binom{2d}{d}_{\#p} = \binom{3d}{d}_{\#p}$,

(b) $\left(\binom{a}{d}\binom{3d-a}{d}\right)_{\#p} \ge \binom{2d}{d}_{\#p}$,

(c) $\left(\binom{a}{d}\binom{3d-1-a}{d}\right)_{\#p} \ge \binom{2d}{d}_{\#p}$, *and*

(d) $\left(\binom{a}{d-1}\binom{3d-2-a}{d}\right)_{\#p} \ge \binom{2d}{d}_{\#p}$

for all integers a with $0 \le a \le 2d$.

The proof of Theorem 1.9 is given in Section 4.

Definition 1.10. The set T_0 is empty and T_2 is the set of all positive integers. Assume that p is an odd prime integer. The non-negative integer a is in T_p if there exists an odd integer J and a power $q = p^e$ of p, with $e \ge 1$, such that

$$\begin{cases} Jq - \frac{q+1}{3} \le a \le Jq + \frac{q-2}{3} & \text{if } q \equiv 2 \mod 3 \\ Jq - \frac{q-1}{3} \le a \le Jq + \frac{q-4}{3} & \text{if } q \equiv 1 \mod 3 \\ Jq - \frac{q}{3} \le a \le Jq + \frac{q}{3} - 1 & \text{if } q \equiv 0 \mod 3. \end{cases} \tag{1.11}$$

Remark 1.12. In the language of (0.3), the display (1.11) is equivalent to

$$Jq - \{\tfrac{q}{3}\} \le a < Jq + \{\tfrac{q}{3}\}.$$

Remark 1.13. The following statements are immediately clear.

(a) If a is odd, then $a \in T_3 \iff |a - Jq| < \frac{q}{3}$ for some odd integer J and power $q = p^e$ of $p = 3$.

(b) If a is odd, then $a \in T_3 \iff a - 1 \in T_3$.

A similar alternate definition of T_p is available for each $p \ge 5$.

Observation 1.14. *Fix a prime integer p with $p \ge 5$.*

(1) *A non-negative even integer a is in T_p if and only if there exists an odd integer J and a power $q = p^e$ of p, with $e \ge 1$, such that $|a - Jq| < \frac{q}{3}$.*

(2) *An odd integer a is in T_p if and only if $a + 1$ is in T_p.*

Proof. We first prove (1). Let a and J be integers with a even and J odd. Suppose that $q \equiv 2 \mod 3$. In this case, $Jq - \frac{q+1}{3}$ is odd; so,

$$Jq - \tfrac{q+1}{3} \le a \le Jq + \tfrac{q-2}{3} \iff Jq - \tfrac{q+1}{3} + 1 \le a \le Jq + \tfrac{q-2}{3} \iff |Jq - a| \le \tfrac{q-2}{3}.$$

However, $|Jq-a|$ and $\frac{q-2}{3}$ are both integers and there are no integers in the open interval $(\frac{q-2}{3}, \frac{q}{3})$; so,

$$Jq - \tfrac{q+1}{3} \le a \le Jq + \tfrac{q-2}{3} \iff |Jq-a| < \tfrac{q}{3}.$$

Now suppose that $q \equiv 1 \mod 3$. In this case, $Jq - \frac{q-1}{3}$ is odd; so,

$$Jq - \tfrac{q-1}{3} \le a \le Jq + \tfrac{q-4}{3} \iff Jq - \tfrac{q-1}{3} + 1 \le a \le Jq + \tfrac{q-4}{3} \iff |Jq-a| \le \tfrac{q-4}{3}.$$

The integers $|Jq-a|$ and $\frac{q-4}{3}$ are both odd. There are no odd integers in the open interval $(\frac{q-4}{3}, \frac{q}{3})$; so,

$$Jq - \tfrac{q-1}{3} \le a \le Jq + \tfrac{q-4}{3} \iff |Jq-a| < \tfrac{q}{3}.$$

Assertion (2) follows from the fact that both left hand end points $Jq - \frac{q+1}{3}$ (if $q \equiv 2$) and $Jq - \frac{q-1}{3}$ (if $q \equiv 1$) in the definition of T_p are odd integers and both right hand end points are even integers. $\square$

Examples 1.15. We see that

$$\begin{aligned} T_3 &= [2,3] \cup [6,11] \cup [14,15] \cup [18,35] \cup [38,39] \cup [42,47] \cup [50,51] \cup [54,107] \\ &\quad \cup [110,111] \cup [114,119] \cup [122,123] \cup [126,143] \cup [146,147] \cup \dots, \\ T_5 &= [3,6] \cup [13,36] \cup [43,46] \cup [53,56] \cup [63,186] \cup \dots, \text{ and} \\ T_7 &= [5,8] \cup [19,22] \cup [33,64] \cup [75,78] \cup [89,92] \cup [103,106] \cup \dots . \end{aligned}$$

Now that the sets S_c and T_c have been thoroughly introduced, we turn our attention to other concepts which we use throughout the paper. The $s \times s$ matrix $\varphi = (\varphi_{i,j})$ is *alternating* if $\varphi_{j,i} = -\varphi_{i,j}$ and $\varphi_{i,i} = 0$, for all i and j. The *Pfaffian* of φ is

$$\mathrm{Pf}(\varphi) = \begin{cases} 0 & \text{if } s \text{ is odd} \\ 1 & \text{if } s = 0 \\ \varphi_{1,2} & \text{if } s = 2 \\ \sum_{j=2}^{s} (-1)^j \varphi_{1,j} \, \mathrm{Pf}\begin{pmatrix} \varphi \text{ with rows and columns} \\ 1 \text{ and } j \text{ deleted} \end{pmatrix} & \text{if } s \ge 4 \text{ is even.} \end{cases}$$

If s is odd, then, for each index ℓ with $1 \le \ell \le s$, we define $\mathrm{Pf}_\ell(\varphi)$:

$$\mathrm{Pf}_\ell(\varphi) = (-1)^{\ell+1} \, \mathrm{Pf}(\varphi \text{ with row and column } \ell \text{ deleted}). \tag{1.16}$$

It is well-known that the classical adjoint of a square matrix M satisfies $M \,\mathrm{Adj}\, M = (\det M) I$. The corresponding statement for Pfaffians is recorded below.

Definition 1.17. If φ is an $s\times s$ alternating matrix for some positive even inter s, then define $\varphi\check{}$ to be the alternating $s\times s$ matrix with

$$(\varphi\check{})_{i,j}=\begin{cases}(-1)^{i+j}\operatorname{Pf}(\varphi \text{ with rows and columns } i \text{ and } j \text{ deleted}) & \text{if } i<j\\ 0 & \text{if } i=j\\ (-1)^{i+j+1}\operatorname{Pf}(\varphi \text{ with rows and columns } i \text{ and } j \text{ deleted}) & \text{if } j<i.\end{cases}$$

Observation 1.18. *If φ is an $s\times s$ alternating matrix for some positive even integer s, then $\varphi\varphi\check{}=\operatorname{Pf}(\varphi)\cdot I=\varphi\check{}\varphi$.*

The proof is straightforward.

Examples 1.19. 1. If $\varphi=\begin{bmatrix}0&f\\-f&0\end{bmatrix}$, then $\varphi\check{}=\begin{bmatrix}0&-1\\1&0\end{bmatrix}$.
2. If

$$\varphi=\begin{bmatrix}0&\varphi_{1,2}&\varphi_{1,3}&\varphi_{1,4}\\-\varphi_{1,2}&0&\varphi_{2,3}&\varphi_{2,4}\\-\varphi_{1,3}&-\varphi_{2,3}&0&\varphi_{3,4}\\-\varphi_{1,4}&-\varphi_{2,4}&-\varphi_{3,4}&0\end{bmatrix}$$

is an arbitrary 4×4 alternating matrix, then $\varphi\check{}$ is the 4×4 alternating matrix

$$\varphi\check{}=\begin{bmatrix}0&-\varphi_{3,4}&\varphi_{2,4}&-\varphi_{2,3}\\\varphi_{3,4}&0&-\varphi_{1,4}&\varphi_{1,3}\\-\varphi_{2,4}&\varphi_{1,4}&0&-\varphi_{1,2}\\\varphi_{2,3}&-\varphi_{1,3}&\varphi_{1,2}&0\end{bmatrix}.$$

One easily sees that $\varphi\varphi\check{}=\operatorname{Pf}(\varphi)I=\varphi\check{}\varphi$.
3. We are particularly interested in the 4×4 alternating matrices $\varphi_{r,n-r}$. If r and s are non-negative integers, then $\varphi_{r,s}$ is the matrix

$$\varphi_{r,s}=\begin{bmatrix}0&z^r&-y^r&x^s\\-z^r&0&x^r&y^s\\y^r&-x^r&0&z^s\\-x^s&-y^s&-z^s&0\end{bmatrix}\tag{1.20}$$

with entries in $\boldsymbol{k}[x,y,z]$. We observe that, in the sense of Definition 1.17, $\varphi_{r,s}\check{}$ is equal to $-\varphi_{s,r}$. Thus, the matrices $\varphi_{r,s}$ and $-\varphi_{s,r}$ form a matrix factorization of

$$(x^{r+s}+y^{r+s}+z^{r+s})=\operatorname{Pf}(\varphi_{r,s})=\operatorname{Pf}(\varphi_{s,r}).$$

In particular, the matrices $\varphi_{r,n-r}$ and $-\varphi_{n-r,r}$ form a matrix factorization of the defining equation of the diagonal hypersurface ring $R_{\boldsymbol{k},n}$; that is,

$$\varphi_{r,n}\varphi_{n-r,n}=-(x^n+y^n+z^n)I,\tag{1.21}$$

where I is the 4×4 identity matrix.

There is a curious identity which relates the maximal order Pfaffians of an odd-sized alternating matrix to the Pfaffians of alternating matrices made from its constituent pieces. Remarkably enough, this identity plays a critical role in our calculations. The matrix $\varphi^{\vee}$ is defined in Definition 1.17. We write $\hat{a}$ to indicate that the entry a has been deleted. The function Pf_ℓ is defined in (1.16).

Observation 1.22. *Let d_2 be a $(m+3) \times (m+3)$ alternating matrix with m even. Partition d_2 into submatrices*

$$d_2 = \begin{bmatrix} \varphi & \psi^{\mathrm{T}} \\ -\psi & \Phi \end{bmatrix},$$

where φ is a $m \times m$ alternating matrix, Φ is a 3×3 alternating matrix, and ψ is a $3 \times m$ matrix. Then for each index ℓ, with $1 \le \ell \le 3$,

$$\mathrm{Pf}_{m+\ell}(d_2) = \mathrm{Pf}_\ell(\psi \varphi^{\vee} \psi^{\mathrm{T}}) + \mathrm{Pf}(\varphi) \cdot \mathrm{Pf}_\ell(\Phi).$$

Proof. We prove the result for $\ell = 1$. Expand down column $m+3$ to see that $\mathrm{Pf}_{m+1}(d_2)$ is

$$\sum_{j=1}^{m} (-1)^{j+1} \psi_{3,j} \,\mathrm{Pf} \begin{pmatrix} 0 & \dots & \widehat{\varphi_{1,j}} & \dots & \varphi_{1,m} & \psi_{2,1} \\ -\varphi_{1,2} & \dots & \widehat{\varphi_{2,j}} & \dots & \varphi_{2,m} & \psi_{2,2} \\ \vdots & & \vdots & & \vdots & \vdots \\ \widehat{-\varphi_{1,j}} & \dots & \hat{0} & \dots & \widehat{\varphi_{2,j}} & \widehat{\psi_{2,j}} \\ \vdots & & \vdots & & \vdots & \vdots \\ -\varphi_{1,m} & \dots & \widehat{-\varphi_{j,m}} & \dots & 0 & \psi_{2,m} \\ -\psi_{2,1} & \dots & \widehat{-\psi_{2,j}} & \dots & -\psi_{2,m} & 0 \end{pmatrix} + \Phi_{2,3} \,\mathrm{Pf}(\varphi).$$

Expand down the column with entries $\{\psi_{2,*}\}$ to see that $\mathrm{Pf}_{m+1}(d_2)$ is

$$= \sum_{j=1}^{m} (-1)^{j+1} \psi_{3,j} \sum_{i=0}^{m} \sigma_{i,j} \psi_{2,i} \,\mathrm{Pf} \begin{pmatrix} \varphi \text{ with rows and columns} \\ i \text{ and } j \text{ deleted} \end{pmatrix} + \Phi_{2,3} \,\mathrm{Pf}(\varphi),$$

where

$$\sigma_{i,j} = \begin{cases} (-1)^{i+1} & \text{if } i < j \\ 0 & \text{if } i < j \\ (-1)^{i} & \text{if } j < i. \end{cases}$$

Let D be the 3×3 alternating matrix $\psi \varphi^{\vee} \psi^{\mathrm{T}} + \mathrm{Pf}(\varphi) \cdot \Phi$. Observe that $\mathrm{Pf}_{m+1}(d_2)$ is equal to the entry of D in row 2 and column 3. On the other hand, $D_{2,3} = \mathrm{Pf}_1(D)$. Thus, $\mathrm{Pf}_{m+1}(d_2) = \mathrm{Pf}_1(D)$, and the proof is complete. $\square$

Definition 1.23. If Q is a noetherian artinian graded $\boldsymbol{k}$-algebra with unique homogeneous maximal ideal Q_+, then the socle of Q,

$$\operatorname{soc} Q = 0 : Q_+ = \{q \in Q \mid qQ_+ = 0\},$$

is a finite dimensional graded $\boldsymbol{k}$-vector space: $\operatorname{soc} Q = \bigoplus_{i=1}^{\ell} \boldsymbol{k}(-d_i)$. The numbers $d_1 \le d_2 \le \cdots \le d_\ell$ are the *socle degrees of* Q. When recording socle degrees, we use the convention $d_j : r$ to indicate that $d_j = d_{j+1} = \cdots = d_{j+r-1}$.

Remark 1.24. If the ring Q of Definition 1.23 is the quotient of a polynomial ring $\boldsymbol{k}[x_1, \dots, x_m]$, then one may read the socle degrees of Q from a minimal homogeneous resolution of Q by free P-modules. If

$$0 \to \bigoplus_{i=1}^{\ell} P(-b_i) \to \cdots \to P \to Q$$

is such a resolution (with $b_1 \le \cdots \le b_\ell$) and $\deg x_i = 1$ for all i, then the socle degrees of Q are

$$b_1 - m \le \cdots \le b_\ell - m;$$

see, for example [8, Cor. 1.7].

If M is an R-module and

$$\mathbb{F}: \quad \dots \xrightarrow{d_{i+1}} F_i \xrightarrow{d_i} F_{i-1} \xrightarrow{d_{i-1}} \dots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \to M \to 0$$

is a minimal resolution of M, then the i^{th}-syzygy of M is $\operatorname{syz}_i M = \operatorname{im} d_i$. In particular, the truncation

$$\mathbb{F}_{\ge i}: \quad \dots \xrightarrow{d_{i+2}} F_{i+1} \xrightarrow{d_{i+1}} F_i \tag{1.25}$$

of $\mathbb{F}$ is a minimal resolution of $\operatorname{syz}_i M$.

Some of our calculations involve binomial coefficients. We recall that $\binom{m}{i}$ makes sense for any pair of integers m and i; furthermore we recall the standard properties of these objects.

Definition 1.26. For integers i and m, the binomial coefficient $\binom{m}{i}$ is defined to be

$$\binom{m}{i} = \begin{cases} \dfrac{m(m-1)\cdots(m-i+1)}{i!} & \text{if } 0 < i, \\ 1 & \text{if } 0 = i, \text{ and} \\ 0 & \text{if } i < 0. \end{cases}$$

Facts 1.27.

(a) *If i and m are integers with $0 \le m < i$, then* $\binom{m}{i} = 0$.
(b) *If i and m are integers, then* $\binom{m}{i-1} + \binom{m}{i} = \binom{m+1}{i}$.
(c) *If i and m are integers with $0 \le m$, then* $\binom{m}{i} = \binom{m}{m-i}$.
(d) *If i and m are integers, then* $(m-i)\binom{m}{i} = \binom{m}{i+1}(i+1)$.
(e) *If i and m are integers, then* $i\binom{m}{i} = \binom{m-1}{i-1}m$.
(f) *If $0 \le i \le m$ are integers, then* $\binom{m}{i} = \frac{m!}{i!(m-i)!}$.

Section 2. The technique for resolving $Q_{\boldsymbol{k},n,N}$.

Lemma 2.3 describes our main technique for resolving $Q_{\boldsymbol{k},n,N}$ as a module over $R_{\boldsymbol{k},n}$. The Lemma is set in a slightly more general context. To apply the Lemma in the context of $Q_{\boldsymbol{k},n,N}$, one takes $P = \boldsymbol{k}[x,y,z]$, $m = 4$,

$$\boldsymbol{x} = [\, x^N \quad y^N \quad z^N \,], \qquad f = x^n + y^n + z^n,$$
$$X = \begin{bmatrix} 0 & z^N & -y^N \\ -z^N & 0 & x^N \\ y^N & -x^N & 0 \end{bmatrix}, \quad \text{and} \quad \varphi = \begin{cases} \varphi_{r,n-r} & \text{if } \theta \text{ is odd, or} \\ \varphi_{n-r,r} & \text{if } \theta \text{ is even,} \end{cases} \tag{2.1}$$

where we have retained the notation $N = \theta n + r$, with $0 \le r \le n-1$. Recall from (1.21) that $\operatorname{Pf}(\varphi_{r,n-r}) = \operatorname{Pf}(\varphi_{n-r,r}) = f$ and $\varphi_{r,n-r}\varphi_{n-r,r} = -fI$. To apply Lemma 2.3, one must find a matrix ψ and a unit u so that

$$\text{the entries of } \psi\varphi^{\check{}}\psi^{\mathrm{T}} - uX \text{ are in the ideal } (f)P. \tag{2.2}$$

(The matrix $\varphi^{\check{}}$ is defined in Definition 1.17.) As soon as (2.2) is accomplished, then the $R_{\boldsymbol{k},n}$-resolution of $Q_{\boldsymbol{k},n,N}$ is given in conclusion (3) of Lemma 2.3. Conclusions (1) and (2) are steps along the way to conclusion (3); these steps are of interest in their own right. Conclusion (1) lists the generators of $(x^N, y^N, z^N) : f$ and conclusion (2) gives the P-resolution of $Q_{\boldsymbol{k},n,N}$. We can read the socle degrees of $Q_{\boldsymbol{k},n,N}$ and, indeed, the entire Hilbert function of $Q_{\boldsymbol{k},n,N}$, from the resolution of conclusion (2). On the other hand, it is difficult to find ψ and u so that (2.2) is satisfied. We show our solutions ψ and u to the problem posed in (2.2) in the proof of Theorem 3.5. We prove that our solutions work in Section 7.

Lemma 2.3. *Let P be a commutative noetherian ring, $\boldsymbol{x} = [x_1, x_2, x_3]$ be a 1×3 matrix whose entries generate a perfect grade three ideal in P, m be a positive even integer, φ be an $m \times m$ alternating matrix with entries in P, ψ be a $3 \times m$ matrix with entries in P, and u be a unit of P. Define f to be the Pfaffian of φ and X to be the 3×3 alternating matrix with $\operatorname{Pf}_i(X) = x_i$ for $1 \le i \le 3$. Assume that the* *entries of* $\psi\check{\varphi}\psi^{\mathrm{T}} - uX$ *are in the ideal* $(f)P$. *Define* Φ *to be a* 3×3 *alternating matrix with*

$$\psi\check{\varphi}\psi^{\mathrm{T}} + f\Phi = uX, \tag{2.4}$$

and define

$$d_2 = \begin{bmatrix} \varphi & \psi^{\mathrm{T}} \\ -\psi & \Phi \end{bmatrix}. \tag{2.5}$$

Let R *be the ring* $P/(f)$ *and* Q *be the ring* $P/(f, I_1(\boldsymbol{x}))$. *Then the following statements hold.*

(1) *The ideal* $I_1(\boldsymbol{x}) : f$ *of* P *is generated by the maximal order Pfaffians of* d_2.
(2) *The maps and modules*

$$\mathbb{F}: \quad 0 \to F_3 \xrightarrow{f_3} F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0,$$

with

$$F_3 = P^m, \quad F_2 = \begin{matrix} P^m \\ \oplus \\ P^3 \end{matrix}, \quad F_1 = \begin{matrix} P^3 \\ \oplus \\ P \end{matrix}, \quad F_0 = P,$$

$$f_3 = \begin{bmatrix} \varphi \\ -u\psi \end{bmatrix}, \quad f_2 = \begin{bmatrix} u\psi\check{\varphi} & fI \\ -\boldsymbol{b} & -\boldsymbol{x} \end{bmatrix}, \quad f_1 = [\,\boldsymbol{x} \quad f\,]$$

form a resolution of Q *by free* P*-modules, where*

$$\boldsymbol{b} = [\,\mathrm{Pf}_1(d_2) \quad \dots \quad \mathrm{Pf}_m(d_2)\,].$$

(3) *The maps and modules*

$$\dots \to R^m \xrightarrow{\varphi} R^m \xrightarrow{\check{\varphi}} R^m \xrightarrow{\varphi} R^m \xrightarrow{\psi\check{\varphi}} R^3 \xrightarrow{\boldsymbol{x}} R$$

form a resolution of Q *by free* R*-modules.*

Proof. The ideal $I_1(\boldsymbol{x})$ is generated by a regular sequence; hence, the Koszul complex:

$$0 \to P \xrightarrow{\boldsymbol{x}^{\mathrm{T}}} P^3 \xrightarrow{\boldsymbol{X}} P^3 \xrightarrow{\boldsymbol{x}} P \tag{2.6}$$

is a resolution of $P/I_1(\boldsymbol{x})$ by free P-modules. Let J be the ideal of P generated by the maximal order Pfaffians of d_2. We will show that J is a grade three Gorenstein ideal and we will compute that $\frac{I_1(\boldsymbol{x}):J}{I_1(\boldsymbol{x})}$ is a cyclic module generated by the class of f. The theory of linkage then shows that $I_1(\boldsymbol{x}):f=J$.

In Observation 1.22 we calculated that the Pfaffians

$$[\,\mathrm{Pf}_{m+1}(d_2)\quad \mathrm{Pf}_{m+2}(d_2)\quad \mathrm{Pf}_{m+3}(d_2)\,]$$

of d_2 are precisely the same as the Pfaffians

$$[\,\mathrm{Pf}_1(\psi\check{\varphi}\psi^{\mathrm{T}}+(\mathrm{Pf}\,\varphi)\Phi)\quad \mathrm{Pf}_2(\psi\check{\varphi}\psi^{\mathrm{T}}+(\mathrm{Pf}\,\varphi)\Phi)\quad \mathrm{Pf}_3(\psi\check{\varphi}\psi^{\mathrm{T}}+(\mathrm{Pf}\,\varphi)\Phi)\,]$$

of the alternating matrix $\psi\check{\varphi}\psi^{\mathrm{T}}+(\mathrm{Pf}\,\varphi)\Phi$. The hypothesis (2.4) ensures that $\psi\check{\varphi}\psi^{\mathrm{T}}+(\mathrm{Pf}\,\varphi)\Phi=uX$; hence,

$$[\,\mathrm{Pf}_{m+1}(d_2)\quad \mathrm{Pf}_{m+2}(d_2)\quad \mathrm{Pf}_{m+3}(d_2)\,]=u\boldsymbol{x}$$

and therefore J, which is the ideal of maximal order Pfaffians of d_2, has grade three and is a grade three Gorenstein ideal; see, for example [3, Thm. 2.1]. Let $\boldsymbol{b}$ be the matrix defined in the statement of (2). The resolution of P/J by free P-modules is

$$0\to P\xrightarrow{d_3}P^{m+3}\xrightarrow{d_2}P^{m+3}\xrightarrow{d_1}P, \tag{2.7}$$

with

$$d_3=\begin{bmatrix}\boldsymbol{b}^{\mathrm{T}}\\ u\boldsymbol{x}^{\mathrm{T}}\end{bmatrix},\quad d_2 \text{ given in (2.5)},\quad \text{and}\quad d_1=[\,\boldsymbol{b}\quad u\boldsymbol{x}\,].$$

Let $\pi: P/I_1(\boldsymbol{x})\to P/J$ be the natural quotient map. We claim that

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & P & \xrightarrow{u\boldsymbol{x}^{\mathrm{T}}} & P^3 & \xrightarrow{u\boldsymbol{X}} & P^3 & \xrightarrow{u\boldsymbol{x}} & P & \longrightarrow & P/I_1(\boldsymbol{x})\\
 & & {\scriptstyle\alpha_3}\big\downarrow & & \big\downarrow{\scriptstyle\alpha_2} & & \big\downarrow{\scriptstyle\alpha_1} & & {\scriptstyle=}\big\downarrow & & {\scriptstyle\pi}\big\downarrow\\
0 & \longrightarrow & P & \xrightarrow{d_3} & P^{m+3} & \xrightarrow{d_2} & P^{m+3} & \xrightarrow{d_1} & P & \longrightarrow & P/J,
\end{array} \tag{2.8}$$

with

$$\alpha_3=f,\quad \alpha_2=\begin{bmatrix}-\check{\varphi}\psi^{\mathrm{T}}\\ fI\end{bmatrix},\quad \alpha_1=\begin{bmatrix}0_{4\times 3}\\ I_{3\times 3}\end{bmatrix},$$

is a map of complexes from a resolution of $P/I_1(\boldsymbol{x})$ isomorphic to (2.6) to the resolution (2.7) of P/J. To see that $d_2\alpha_2=\alpha_1X$, one uses the fact that $\varphi\check{\varphi}=fI$ (see Observation 1.18) and the hypothesis (2.4). To see that $d_3\alpha_3=\alpha_2(u\boldsymbol{x}^{\mathrm{T}})$, observe first that

$$d_2\alpha_2(u\boldsymbol{x}^{\mathrm{T}})=\alpha_1(u\boldsymbol{X})(u\boldsymbol{x}^{\mathrm{T}})=0.$$

Thus, $\alpha_2(u\boldsymbol{x}^{\mathrm{T}})$ is in the kernel of d_2. The complex (2.7) is exact; so, $\alpha_2(u\boldsymbol{x}^{\mathrm{T}}) = vd_3$ for some v in P. On the other hand,

$$\alpha_2(u\boldsymbol{x}^{\mathrm{T}}) = \begin{bmatrix} -\varphi\check{}\,\psi^{\mathrm{T}}(u\boldsymbol{x}^{\mathrm{T}}) \\ fu\boldsymbol{x}^{\mathrm{T}} \end{bmatrix} \quad \text{and} \quad vd_3 = v\begin{bmatrix} \boldsymbol{b}^{\mathrm{T}} \\ u\boldsymbol{x}^{\mathrm{T}} \end{bmatrix}.$$

It is obvious that v must be f and (2.8) is indeed a map of complexes.

Standard results from homological algebra now show that $I_1(\boldsymbol{x})\colon f$ is equal to J. Indeed, the long exact sequence of homology which is obtained by applying $\operatorname{Hom}_P(\_\_, P)$ to the short exact sequence

$$0 \to J/I_1(\boldsymbol{x}) \to P/I_1(\boldsymbol{x}) \xrightarrow{\pi} P/J \to 0,$$

yields that

$$\pi^*\colon\ \operatorname{Ext}^3_P(P/J, P) \to \operatorname{Ext}^3_P(P/I_1(\boldsymbol{x}), P) \tag{2.9}$$

is an injection. On the other hand, one may apply $\operatorname{Hom}_P(\_\_, P)$ to the comparison map of resolutions which is given in (2.8) to see that the injection (2.9) is also equal to

$$P/J \xrightarrow{f} P/I_1(\boldsymbol{x}).$$

In other words, $I_1(\boldsymbol{x})\colon f$ is equal to J. This establishes (1). Take the mapping cone of the dual of (2.8) to establish (2).

We prove (3). Let $\overline{}$ represent the functor $\_\_ \otimes_P R$. Take the resolution $\mathbb{F}$ of Q from (2). We see that $\overline{\mathbb{F}}$ is a complex of free R-modules with

$$\mathrm{H}(\overline{\mathbb{F}}) = \operatorname{Tor}_i^P(R, Q) = \begin{cases} Q & \text{if } i = 0 \text{ or } 1 \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, the cycle

$$\xi = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{2.10}$$

in $\overline{F}_1$ represents a generator of $\mathrm{H}_1(\overline{\mathbb{F}})$. We kill the homology in $\overline{\mathbb{F}}$. Define R-module homomorphisms $\beta_i\colon \overline{F}_i \to \overline{F}_{i+1}$ by

$$\beta_2 = [\,\varphi\check{}\quad 0\,], \quad \beta_1 = \begin{bmatrix} 0 & 0 \\ -I & 0 \end{bmatrix}, \quad \beta_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

A straightforward calculation shows that

$$\begin{tikzcd}[column sep=small] 0 \arrow[r] & \overline{F}_3 \arrow[r, "\overline{f}_3"] & \overline{F}_2 \arrow[r, "\overline{f}_2"] \arrow[d, "\beta_2"'] & \overline{F}_1 \arrow[r, "\overline{f}_1"] \arrow[d, "\beta_1"'] & \overline{F}_0 \arrow[d, "\beta_0"'] \\ & 0 \arrow[r] & \overline{F}_3 \arrow[r, "\overline{f}_3"] & \overline{F}_2 \arrow[r, "\overline{f}_2"] & \overline{F}_1 \arrow[r, "\overline{f}_1"] & \overline{F}_0 \end{tikzcd} \tag{2.11}$$

is a map of complexes. It is clear that β_0 induces an isomorphism from H_0 of the top line of (2.11) to H_1 of the bottom line of (2.11). Let $\mathbb{M}$ be the total complex of (2.11). We have shown that the homology of $\mathbb{M}$ is concentrated in positions 0 and 3 and the ξ from (2.10) of the summand $\overline{F}_1$ in $\mathbb{M}_3 = \overline{F}_1 \oplus \overline{F}_3$ represents a generator of $\mathrm{H}_3(\mathbb{M})$. Iterate this process to see that the Q is resolved by the total complex $\mathbb{T}$ of the following infinite double complex:

$$\begin{tikzcd}[column sep=small] & \vdots \arrow[d, "\beta_2"'] & \vdots \arrow[d, "\beta_1"'] & \vdots \arrow[d, "\beta_0"'] \\ 0 \arrow[r] & \overline{F}_3 \arrow[r, "\overline{f}_3"] & \overline{F}_2 \arrow[r, "\overline{f}_2"] \arrow[d, "\beta_2"'] & \overline{F}_1 \arrow[r, "\overline{f}_1"] \arrow[d, "\beta_1"'] & \overline{F}_0 \arrow[d, "\beta_0"'] \\ & 0 \arrow[r] & \overline{F}_3 \arrow[r, "\overline{f}_3"] & \overline{F}_2 \arrow[r, "\overline{f}_2"] \arrow[d, "\beta_2"'] & \overline{F}_1 \arrow[r, "\overline{f}_1"] \arrow[d, "\beta_1"'] & \overline{F}_0 \arrow[d, "\beta_0"'] \\ & & 0 \arrow[r] & \overline{F}_3 \arrow[r, "\overline{f}_3"] & \overline{F}_2 \arrow[r, "\overline{f}_2"] & \overline{F}_1 \arrow[r, "\overline{f}_1"] & \overline{F}_0. \end{tikzcd}$$

The complex of (3) is a summand of $\mathbb{T}$. $\square$

Corollary 2.12. *If the ring $Q_{\boldsymbol{k},n,N}$ of* (0.2) *has finite projective dimension over $R_{\boldsymbol{k},n}$, then the ideal $(x^N, y^N, z^N) : (x^n + y^n + z^n)$ in $P = \boldsymbol{k}[x, y, z]$ can be generated by* 5 *elements.*

Proof. Let $R = R_{\boldsymbol{k},n}$ and $Q = Q_{\boldsymbol{k},n,N}$. Assume $\operatorname{pd}_R Q < \infty$. The rings R and Q have depth 2 and 0, respectively. The Auslander-Buchsbaum Theorem guarantees that $\operatorname{pd}_R Q = 2$ and therefore, the Hilbert-Burch Theorem ensures that there exists a 3×2 matrix $\bar{\psi}$, over R, whose signed maximal order minors are x^N, y^N, z^N. (See (5.2), if necessary.) Lift $\bar{\psi}$ back to a matrix ψ in P. Let φ be the matrix $\begin{bmatrix} 0 & f \\ -f & 0 \end{bmatrix}$ over P, where $f = x^n + y^n + z^n$. Recall from Example 1.19 that $\varphi^{\check{}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Observe that the non-zero entries of $\psi\check{\varphi}\psi^{\mathrm{T}}$ are the maximal order minors of ψ, up to sign. Indeed, if $u=-1$ and

$$X=\begin{bmatrix} 0 & z^N & -y^N \\ -z^N & 0 & x^N \\ y^N & -x^N & 0 \end{bmatrix},$$

then entries of of the alternating matrix $\psi\check{\varphi}\psi^{\mathrm{T}}-uX$ are in the ideal $(f)P$. The critical hypothesis (2.2) of Lemma 2.3 is satisfied; hence, the ideal $(x^N,y^N,z^N):f$ of P if generated by the maximal order Pfaffians of the 5×5 alternating matrix which is produced in Lemma 2.3. $\square$

Remark 2.13. The ideal $(x^N,y^N,z^N):f$ was introduced in (0.4). It is shown in Theorem 6.2 that $\operatorname{pd}_R Q=\infty$ if and only if the hypotheses of Theorem 3.5 hold. When the hypotheses of Theorem 3.5 hold, then the ideal (0.4) is 7-generated. When Theorems 6.2 and 3.5 are combined with Corollary 2.12, we see that $\operatorname{pd}_R Q$ is infinite if and only if the ideal of (0.4) is 7-generated.

SECTION 3. THE RESOLUTIONS OF $Q_{\boldsymbol{k},n,N}$ WHEN $\lfloor \frac{N}{n}\rfloor$ IS IN S_c.

The following notation and hypotheses are in effect throughout this section. Recall the set S_c from Definition 1.1.

Data 3.1. Let $\boldsymbol{k}$ be a field of characteristic c, n and N be positive integers with n not dividing N. Write $N=\theta n+r$ for integers θ and r with $0<r<n$. Let P be the polynomial ring $P=\boldsymbol{k}[x,y,z]$, R be the diagonal hypersurface ring $R_{\boldsymbol{k},n}$ of (0.1), and Q be the quotient ring $Q_{\boldsymbol{k},n,N}$ of (0.2). **Assume that θ is in the set S_c.**

In Theorem 3.5 we give the minimal homogeneous resolution of Q by free P-modules and also the minimal homogeneous resolution of Q by free R-modules. Our proof is based on the method of Lemma 2.3. One must find a matrix ψ and a unit u so that (2.2) is satisfied. We use the polynomials $\operatorname{Poly}_{d,a,b}$ of Definition 3.2 to build our candidate for (ψ,u). The key calculation which shows that our candidate for (ψ,u) satisfies (2.2) is called Lemma 3.4. The proof of Lemma 3.4 is given in Section 7.

Definition 3.2. Given non-negative integers d, a, b, we define $\operatorname{Poly}_{d,a,b}(A,B)$ to be the following polynomial in the polynomial ring $\mathbb{Z}[A,B]$:

$$\operatorname{Poly}_{d,a,b}(A,B)=\sum_{i=0}^{d}(-1)^i\binom{a+d-i}{a}\binom{b+i}{b}A^{d-i}B^i.$$

Remark. We see that $\operatorname{Poly}_{d,b,a}(B,A)=(-1)^d\operatorname{Poly}_{d,a,b}(A,B)$.

Definition 3.3. For each positive integer δ define the polynomials $\mathfrak{P}_{2\delta-1}$ and $\mathfrak{P}_{2\delta}$ in $\mathbb{Z}[A,B,C]$ by

$$\mathfrak{P}_{2\delta-1}(A,B,C)=\begin{cases}(-1)^{\delta}A\operatorname{Poly}_{\delta-1,\delta,\delta-1}(A,B)\operatorname{Poly}_{\delta-1,\delta,\delta-1}(A,C)\\ +B\operatorname{Poly}_{\delta-1,\delta,\delta-1}(B,A)\operatorname{Poly}_{\delta-1,\delta,\delta-1}(A,C)\\ +C\operatorname{Poly}_{\delta-1,\delta,\delta-1}(A,B)\operatorname{Poly}_{\delta-1,\delta,\delta-1}(C,A)\end{cases}$$

and

$$\mathfrak{P}_{2\delta}(A,B,C)=\begin{cases}(-1)^{\delta+1}\operatorname{Poly}_{\delta,\delta,\delta}(A,B)\operatorname{Poly}_{\delta,\delta,\delta}(A,C)\\ +B\operatorname{Poly}_{\delta-1,\delta,\delta}(B,A)\operatorname{Poly}_{\delta,\delta,\delta}(A,C)\\ +C\operatorname{Poly}_{\delta,\delta,\delta}(A,B)\operatorname{Poly}_{\delta-1,\delta,\delta}(C,A).\end{cases}$$

Remark. We see that

$$\mathfrak{P}_{2\delta-1}(A,B,C)=\mathfrak{P}_{2\delta-1}(A,C,B)\quad\text{and}\quad\mathfrak{P}_{2\delta}(A,B,C)=\mathfrak{P}_{2\delta}(A,C,B).$$

Lemma 3.4. *For each positive integer δ, the polynomials*

$$P_{2\delta-1}(A,B,C)=\mathfrak{P}_{2\delta-1}(A,B,C)+(-1)^{\delta+1}\binom{2\delta}{\delta}\binom{3\delta-1}{\delta-1}A^{2\delta-1}$$

and

$$P_{2\delta}(A,B,C)=\mathfrak{P}_{2\delta}(A,B,C)+(-1)^{\delta}\binom{2\delta}{\delta}\binom{3\delta}{\delta}A^{2\delta}$$

are in the ideal $(A+B+C)\mathbb{Z}[A,B,C]$.

Theorem 3.5. *Assume that the notation and hypotheses of Data* 3.1 *are in effect. Then the following statements hold.*

(1) *If $\theta=2\delta-1$ is odd, then the minimal homogeneous resolution of Q by free P-modules has the form*

$$0\to\begin{array}{c}P(-3\delta n+n-2r)^3\\ \oplus\\ P(-3\delta n)\end{array}\to\begin{array}{c}P(-3\delta n+n-r)^3\\ \oplus\\ P(-3\delta n+2n-3r)\\ \oplus\\ P(-2\delta n-r)^3\end{array}\to\begin{array}{c}P(-N)^3\\ \oplus\\ P(-n)\end{array}\to P$$

and the minimal homogeneous resolution of Q by free R-modules has the form

$$\cdots\xrightarrow{\varphi_{r,n-r}}\begin{array}{c}R(-3\delta n-r)^3\\ \oplus\\ R(-3\delta n+n-3r)\end{array}\xrightarrow{\varphi_{n-r,r}}\begin{array}{c}R(-3\delta n+n-2r)^3\\ \oplus\\ R(-3\delta n)\end{array}$$

$$\xrightarrow{\varphi_{r,n-r}}\begin{array}{c}R(-3\delta n+n-r)^3\\ \oplus\\ R(-3\delta n+2n-3r)\end{array}\to R(-N)^3\to R.$$

In particular, $\operatorname{pd}_R Q = \infty$ *and the socle degrees of* Q *are*

$$(3\delta n - n + 2r - 3){:}3, \quad 3\delta n - 3.$$

(2) *If* $\theta = 2\delta$ *is even, then the minimal homogeneous resolution of* Q *by free* P*-modules has the form*

$$0 \to \begin{matrix} P(-3\delta n - n - r)^3 \\ \oplus \\ P(-3\delta n - 3r) \end{matrix} \to \begin{matrix} P(-3\delta n - 2r)^3 \\ \oplus \\ P(-3\delta n - n) \\ \oplus \\ P(-2\delta n - n - r)^3 \end{matrix} \to \begin{matrix} P(-N)^3 \\ \oplus \\ P(-n) \end{matrix} \to P$$

and the minimal homogeneous resolution of Q *by free* R*-modules has the form*

$$\dots \xrightarrow{n-r,r} \begin{matrix} R(-3\delta n - n - 2r)^3 \\ \oplus \\ R(-3\delta n - 2n) \end{matrix} \xrightarrow{\varphi_{r,n-r}} \begin{matrix} R(-3\delta n - n - r)^3 \\ \oplus \\ R(-3\delta n - 3r) \end{matrix}$$

$$\xrightarrow{\varphi_{n-r,r}} \begin{matrix} R(-3\delta n - 2r)^3 \\ \oplus \\ R(-3\delta n - n) \end{matrix} \to R(-N)^3 \to R.$$

In particular, $\operatorname{pd}_R Q = \infty$ *and the socle degrees of* Q *are*

$$(3\delta n + n + r - 3){:}3, \quad 3\delta n + 3r - 3.$$

Remark. All of the differentials in the resolutions of Theorem 3.5 are explicitly described in Lemma 2.3.

Before we prove Theorem 3.5, we observe that this Theorem contains one direction of Theorem 6.2.

Corollary 3.6. *Let* $\boldsymbol{k}$ *be a field of characteristic* c *and let* n *and* N *be positive integers. If* $N = \theta n + r$, *with* $\theta \in S_c$ *and* $1 \le r \le n-1$, *then* $\operatorname{pd}_R Q = \infty$.

We also offer the following reformulation of part of Theorem 3.5

Corollary 3.7. *If the hypotheses of Data 3.1 are in effect, then there exist an exact sequence of* R*-modules*:

$$0 \to M_{\boldsymbol{k},n,r}(-3\delta n + 2n - 2r) \to R(-N)^3 \to R \to Q \to 0, \quad \textit{if } N = (2\delta - 1)n + r,$$

or

$$0 \to M_{\boldsymbol{k},n,n-r}(-3\delta n - r) \to R(-N)^3 \to R \to Q \to 0, \quad \textit{if } N = 2\delta n + r,$$

where the $R_{\boldsymbol{k},n}$-module $M_{\boldsymbol{k},n,r}$ is defined to be the following cokernel:

$$\begin{matrix} R(-n)^3 \\ \oplus \\ R(2r-2n) \end{matrix} \xrightarrow{\varphi_{r,n-r}} \begin{matrix} R(r-n)^3 \\ \oplus \\ R(-r) \end{matrix} \to M_{\boldsymbol{k},n,r} \to 0,$$

and $\varphi_{r,n-r}$ is given in (1.20).

Proof of Theorem 3.5. We apply Lemma 2.3 with $\boldsymbol{x}$, X, f, and φ as given in (2.1) and we consider two cases. In the first case $\operatorname{char} \boldsymbol{k} \neq 3$ and in the second case $\operatorname{char} \boldsymbol{k} = 3$.

We begin with $\operatorname{char} \boldsymbol{k} \neq 3$. Define the integer γ by

$$\gamma = \begin{cases} 1 & \text{if } \operatorname{char} \boldsymbol{k} \text{ equals } 0 \text{ or } 2 \\ p^{\ell} & \text{if } \operatorname{char} \boldsymbol{k} = p \geq 5 \text{ and } \binom{2\delta}{\delta}_{\#p} = \ell. \end{cases}$$

We know from the hypothesis that $\theta \in S_c$. We next show that

$$\binom{2\delta}{\delta}\binom{3\delta}{\delta}/\gamma^2 \text{ is a unit in } \boldsymbol{k}. \tag{3.8}$$

There is nothing to show if $\operatorname{char} \boldsymbol{k} = 0$. If $\operatorname{char} \boldsymbol{k} = 2$, then $\theta \in S_2 = \{0\}$; thus, $\delta = 0$ and again there is nothing to show. If $\operatorname{char} \boldsymbol{k} = p \geq 5$, then Remark 1.2 ensures that $\delta \in D_p$ and Theorem 1.9 yields that $\binom{3\delta}{\delta}_{\#p} = \ell$; and therefore (3.8) holds. We next observe that γ divides every coefficient of each of the polynomials

$$\operatorname{Poly}_{\delta,\delta,\delta}(A,B), \quad \operatorname{Poly}_{\delta-1,\delta,\delta}(A,B), \quad \text{and} \quad \operatorname{Poly}_{\delta-1,\delta,\delta-1}(A,B).$$

(Again, the assertion is obvious if $\operatorname{char} \boldsymbol{k}$ is equal to 0 or 2 and the assertion follows from Theorem 1.9 if $\operatorname{char} \boldsymbol{k} = p \geq 5$.) Let

$$\begin{aligned} \operatorname{Poly}'_{\delta-1,\delta,\delta-1} &= \operatorname{Poly}_{\delta-1,\delta,\delta-1}/\gamma \quad & \operatorname{Poly}'_{\delta,\delta,\delta} &= \operatorname{Poly}_{\delta,\delta,\delta}/\gamma, \\ \operatorname{Poly}'_{\delta-1,\delta,\delta} &= \operatorname{Poly}_{\delta-1,\delta,\delta}/\gamma, & \mathfrak{P}'_{2\delta-1} &= \mathfrak{P}_{2\delta-1}/\gamma^2, \text{ and} \\ \mathfrak{P}'_{2\delta} &= \mathfrak{P}_{2\delta}/\gamma^2 \end{aligned}$$

in $\mathbb{Z}[A,B,C]$ and

$$u_{2\delta-1} = (-1)^{\delta+1}\binom{2\delta}{\delta}\binom{3\delta-1}{\delta-1}/\gamma^2 \quad \text{and} \quad u_{2\delta} = (-1)^{\delta}\binom{2\delta}{\delta}\binom{3\delta}{\delta}/\gamma^2.$$

in $\mathbb{Z}$. Recall that $\binom{3\delta-1}{\delta-1}=\frac{1}{3}\binom{3\delta}{\delta}$ and 3 is a unit in $\boldsymbol{k}$; so $u_{2\delta-1}$ and $u_{2\delta}$ both are units in $\boldsymbol{k}$. We have

$$\begin{aligned}\gamma^2(\mathfrak{P}'_{2\delta-1}(A,B,C)+u_{2\delta-1}A^{2\delta-1}) &= \mathfrak{P}_{2\delta-1}(A,B,C)+(-1)^{\delta+1}\tbinom{2\delta}{\delta}\tbinom{3\delta-1}{\delta-1}A^{2\delta-1}\\ &= P_{2\delta-1}(A,B,C) \text{ and}\end{aligned}$$

$$\gamma^2(\mathfrak{P}'_{2\delta}(A,B,C)+u_{2\delta}A^{2\delta}) = \mathfrak{P}_{2\delta}(A,B,C)+(-1)^{\delta}\tbinom{2\delta}{\delta}\tbinom{3\delta}{\delta}A^{2\delta} = P_{2\delta}(A,B,C).$$

Therefore, according to Lemma 3.4,

$$\gamma^2(\mathfrak{P}'_{2\delta-1}(A,B,C)+u_{2\delta-1}A^{2\delta-1}) \quad\text{and}\quad \gamma^2(\mathfrak{P}'_{2\delta}(A,B,C)+uA^{2\delta})$$

are both in the prime ideal $(A+B+C)\mathbb{Z}[A,B,C]$ of $\mathbb{Z}[A,B,C]$. It follows that

$$\mathfrak{P}'_{2\delta-1}(A,B,C)+u_{2\delta-1}A^{2\delta-1} \quad\text{and}\quad \mathfrak{P}'_{2\delta}(A,B,C)+u_{2\delta}A^{2\delta}$$

are in $(A+B+C)\mathbb{Z}[A,B,C]$. Define $\mathcal{R}_{2\delta-1}(A,B,C)$ and $\mathcal{R}_{2\delta}(A,B,C)$ to be the polynomials in $\mathbb{Z}[A,B,C]$ which satisfy

$$\begin{aligned}\mathfrak{P}'_{2\delta-1}(A,B,C)+u_{2\delta-1}A^{2\delta-1} &= (A+B+C)\mathcal{R}_{2\delta-1}(A,B,C) \text{ and}\\ \mathfrak{P}'_{2\delta}(A,B,C)+u_{2\delta}A^{2\delta} &= (A+B+C)\mathcal{R}_{2\delta}(A,B,C).\end{aligned}\tag{3.9}$$

We now focus on assertion (1); that is, we have $\theta=2\delta-1$ is odd. Recall, from (2.1), that φ is the matrix $\varphi_{r,n-r}$ of (1.20). It follows that $\varphi\check{}$ is the matrix $-\varphi_{n-r,r}$. We see from (1.21) that $\varphi\varphi\check{}=\varphi\check{}\varphi=fI$. Define ψ to be the matrix

$$\begin{bmatrix} 0 & (-1)^{\delta-1}z^r\operatorname{Poly}'_{\delta-1,\delta,\delta-1}(z^n,y^n) & y^r\operatorname{Poly}'_{\delta-1,\delta,\delta-1}(y^n,z^n) & 0\\ z^r\operatorname{Poly}'_{\delta-1,\delta,\delta-1}(z^n,x^n) & 0 & (-1)^{\delta-1}x^r\operatorname{Poly}'_{\delta-1,\delta,\delta-1}(x^n,z^n) & 0\\ (-1)^{\delta-1}y^r\operatorname{Poly}'_{\delta-1,\delta,\delta-1}(y^n,x^n) & x^r\operatorname{Poly}'_{\delta-1,\delta,\delta-1}(x^n,y^n) & 0 & 0\end{bmatrix}$$

and

$$\Phi=\begin{bmatrix} 0 & z^r\mathcal{R}_{2\delta-1}(z^n,x^n,y^n) & -y^r\mathcal{R}_{2\delta-1}(y^n,x^n,z^n)\\ -z^r\mathcal{R}_{2\delta-1}(z^n,x^n,y^n) & 0 & x^r\mathcal{R}_{2\delta-1}(x^n,y^n,z^n)\\ y^r\mathcal{R}_{2\delta-1}(y^n,x^n,z^n) & -x^r\mathcal{R}_{2\delta-1}(x^n,y^n,z^n) & 0\end{bmatrix}.$$

One may easily calculate that $\psi\varphi\check{}\psi^{\mathrm{T}}$ is equal to

$$\begin{bmatrix} 0 & -z^r\mathfrak{P}'_{2\delta-1}(z^n,x^n,y^n) & y^r\mathfrak{P}'_{2\delta-1}(y^n,x^n,z^n)\\ z^r\mathfrak{P}'_{2\delta-1}(z^n,x^n,y^n) & 0 & -x^r\mathfrak{P}'_{2\delta-1}(x^n,y^n,z^n)\\ -y^r\mathfrak{P}'_{2\delta-1}(y^n,x^n,z^n) & x^r\mathfrak{P}'_{2\delta-1}(x^n,y^n,z^n) & 0\end{bmatrix}.$$

Therefore, the definition (3.9) of $\mathcal{R}_{2\delta-1}$ tells us that

$$\psi\varphi\check{}\psi^{\mathrm{T}}+f\Phi=u_{2\delta-1}\boldsymbol{X}.$$

The condition (2.2) is satisfied. Lemma 2.3 applies. All of the entries in all of the matrices are homogeneous of positive degree. Thus, an explicit minimal generating set for the ideal $(x^N, y^N, z^N)\colon f$ of P; an explicit minimal homogeneous resolution of Q by free P-modules, and an explicit minimal homogeneous resolution of Q by free R-modules are all given in Lemma 2.3. (If r had been 0 or n, then Lemma 2.3 would still produce a generating set for $(x^N, y^N, z^N)\colon f$ and free resolutions of Q; however these objects would not be minimal.) To complete the proof of (1), we need only record the degrees of the entries of the matrices in these resolutions. We know that $\deg f = n$, each entry of $\boldsymbol{x}$ has degree N,

$$\deg b_1 = \deg b_2 = \deg b_3 = 3\delta n - 2n + r, \quad \deg b_4 = 3\delta n - 3n + 3r,$$

each entry in columns 1, 2, and 3 of $\psi\varphi\check{}$ has degree δn, each entry of column 4 of $\psi\varphi\check{}$ has degree $\delta n - n + 2r$, each entry of columns 1, 2, and 3 of ψ has degree $\delta n - n + r$, column four of ψ is a zero matrix, and

$$\varphi = \begin{bmatrix} \begin{array}{l} 3\times 3 \text{ matrix with entries} \\ \text{of degree } r \end{array} & \begin{array}{l} 3\times 1 \text{ matrix with entries} \\ \text{of degree } n-r \end{array} \\ \begin{array}{l} 1\times 3 \text{ matrix with entries} \\ \text{of degree } n-r \end{array} & 1\times 1 \text{ zero matrix} \end{bmatrix}.$$

The socle degrees of Q may be read from the P-resolution of Q; see Remark 1.24.

We now focus on assertion (2); that is, we have $\theta = 2\delta$ is even. Recall, from (2.1), that φ is the matrix $\varphi_{n-r,r}$ of (1.20). It follows that $\varphi\check{}$ is the matrix $-\varphi_{r,n-r}$. We still have $\varphi\varphi\check{} = \varphi\check{}\varphi = fI$. Define ψ to be the matrix

$$\begin{bmatrix} \operatorname{Poly}'_{\delta,\delta,\delta}(y^n,z^n) & 0 & 0 & y^rz^r\operatorname{Poly}'_{\delta-1,\delta,\delta}(y^n,z^n) \\ 0 & (-1)^\delta\operatorname{Poly}'_{\delta,\delta,\delta}(x^n,z^n) & 0 & (-1)^{\delta+1}x^rz^r\operatorname{Poly}'_{\delta-1,\delta,\delta}(x^n,z^n) \\ 0 & 0 & \operatorname{Poly}'_{\delta,\delta,\delta}(x^n,y^n) & x^ry^r\operatorname{Poly}'_{\delta-1,\delta,\delta}(x^n,y^n) \end{bmatrix}$$

and

$$\Phi = \begin{bmatrix} 0 & z^r\mathcal{R}_{2\delta}(z^n,x^n,y^n) & -y^r\mathcal{R}_{2\delta}(y^n,x^n,z^n) \\ -z^r\mathcal{R}_{2\delta}(z^n,x^n,y^n) & 0 & x^r\mathcal{R}_{2\delta}(x^n,y^n,z^n) \\ y^r\mathcal{R}_{2\delta}(y^n,x^n,z^n) & -x^r\mathcal{R}_{2\delta}(x^n,y^n,z^n) & 0 \end{bmatrix}.$$

One may easily calculate that $\psi\varphi\check{}\psi^{\mathrm{T}}$ is equal to

$$-\begin{bmatrix} 0 & -z^r\mathfrak{P}'_{2\delta}(z^n,x^n,y^n) & y^r\mathfrak{P}'_{2\delta}(y^n,x^n,z^n) \\ z^r\mathfrak{P}'_{2\delta}(z^n,x^n,y^n) & 0 & -x^r\mathfrak{P}'_{2\delta}(x^n,y^n,z^n) \\ -y^r\mathfrak{P}'_{2\delta}(y^n,x^n,z^n) & x^r\mathfrak{P}'_{2\delta}(x^n,y^n,z^n) & 0 \end{bmatrix}.$$

Therefore, the definition (3.9) of $\mathcal{R}_{2\delta}$ tells us that

$$-\psi\check{\varphi}\psi^{\mathrm{T}} + f\Phi = u_{2\delta}\boldsymbol{X}.$$

The condition (2.2) is satisfied. Lemma 2.3 applies. The proof of (2) is completed just like the proof of (1). We know that $\deg f = n$, each entry of $\boldsymbol{x}$ has degree N,

$$\deg b_1 = \deg b_2 = \deg b_3 = 3\delta n - n + 2r, \quad \deg b_4 = 3\delta n,$$

each entry in columns 1, 2, and 3 of $\psi\check{\varphi}$ has degree $\delta n + r$, each entry of column 4 of $\psi\check{\varphi}$ has degree $\delta n + n - r$, each entry of columns 1, 2, and 3 of ψ has degree δn, each entry in column four of ψ has degree $\delta n - n + 2r$, and

$$\varphi = \begin{bmatrix} \begin{array}{l} 3\times 3 \text{ matrix with entries} \\ \text{of degree } n-r \end{array} & \begin{array}{l} 3\times 1 \text{ matrix with entries} \\ \text{of degree } r \end{array} \\ \begin{array}{l} 1\times 3 \text{ matrix with entries} \\ \text{of degree } r \end{array} & 1\times 1 \text{ zero matrix} \end{bmatrix}.$$

The proof is complete when $\operatorname{char}\boldsymbol{k} \neq 3$. Henceforth, we consider $\operatorname{char}\boldsymbol{k} = 3$. The idea of the proof is exactly the same as in the first case; but all of the details change a little bit. Let

$$d = \begin{cases} \delta & \text{if } \theta \text{ is even} \\ \delta - 1 & \text{if } \theta \text{ is odd.} \end{cases}$$

It follows that

$$\theta = \begin{cases} 2d & \text{if } \theta \text{ is even} \\ 2d+1 & \text{if } \theta \text{ is odd.} \end{cases}$$

The parameter θ is in S_3 by hypothesis; hence, Remark 1.2 shows that $d \in D_3$ and therefore all five statements of part (1) of Theorem 1.9 hold for d.

Define the integers γ and Γ by $\gamma = 3^{\ell}$ and $\Gamma = 3^{L}$ where $\binom{2d}{d}_{\#3} = \ell$ and $\binom{2d+1}{d}_{\#3} = L$. Theorem 1.9 shows that $\binom{3d}{d}_{\#3} = \binom{2d}{d}_{\#3}$; so γ^2 divides $\binom{2d}{d}\binom{3d}{d}$ in $\mathbb{Z}$ and

$$u_{2d} = (-1)^d \binom{2d}{d}\binom{3d}{d}/\gamma^2$$

is a unit in $\boldsymbol{k}$. Theorem 1.9 also guarantees that $\binom{2d+1}{d}_{\#3} = \binom{3d+2}{d}_{\#3}$. Recall that $\binom{2d+2}{d+1} = 2\binom{2d+1}{d}$; so, $\binom{2d+2}{d+1}_{\#3} = \binom{2d+1}{d}_{\#3}$. It follows that Γ^2 divides $\binom{2d+2}{d+1}\binom{3d+2}{d}$ in $\mathbb{Z}$ and

$$u_{2d+1} = (-1)^d \binom{2d+2}{d+1}\binom{3d+2}{d}/\Gamma^2$$

is a unit in $\boldsymbol{k}$. According to (1c) and (1d) from Theorem 1.9, γ divides every coefficient of each of the polynomials

$$\operatorname{Poly}_{d,d,d}(A,B), \quad \operatorname{Poly}_{d-1,d,d}(A,B);$$

furthermore, (1e) from Theorem 1.9 shows that Γ divides every coefficient of the polynomial

$$\operatorname{Poly}_{d,d+1,d}(A,B).$$

Let

$$\begin{aligned} \operatorname{Poly}'_{d,d,d} &= \operatorname{Poly}_{d,d,d}/\gamma & \operatorname{Poly}'_{d,d+1,d} &= \operatorname{Poly}_{d,d+1,d}/\Gamma, \\ \operatorname{Poly}'_{d-1,d,d} &= \operatorname{Poly}_{d-1,d,d}/\gamma, & \mathfrak{P}'_{2d+1} &= \mathfrak{P}_{2d+1}/\Gamma^2, \text{ and} \\ \mathfrak{P}'_{2d} &= \mathfrak{P}_{2d}/\gamma^2 \end{aligned}$$

in $\mathbb{Z}[A,B,C]$. We have

$$\begin{aligned} \Gamma^2(\mathfrak{P}'_{2d+1}(A,B,C) + u_{2d+1}A^{2d+1}) &= \mathfrak{P}_{2d+1}(A,B,C) + (-1)^d \tbinom{2d+2}{d+1}\tbinom{3d+2}{d+1} A^{2d+1} \\ &= P_{2d+1}(A,B,C) \text{ and} \end{aligned}$$

$$\gamma^2(\mathfrak{P}'_{2d}(A,B,C) + u_{2d}A^{2d}) = \mathfrak{P}_{2d}(A,B,C) + (-1)^d\tbinom{2d}{d}\tbinom{3d}{d}A^{2d} = P_{2d}(A,B,C).$$

Therefore, according to Lemma 3.4,

$$\Gamma^2(\mathfrak{P}'_{2d+1}(A,B,C) + u_{2d+1}A^{2d+1}) \quad \text{and} \quad \gamma^2(\mathfrak{P}'_{2d}(A,B,C) + u_{2d}A^{2d})$$

are both in the prime ideal $(A+B+C)\mathbb{Z}[A,B,C]$ of $\mathbb{Z}[A,B,C]$. It follows that

$$\mathfrak{P}'_{2d+1}(A,B,C) + u_{2d+1}A^{2d+1} \quad \text{and} \quad \mathfrak{P}'_{2d}(A,B,C) + u_{2d}A^{2d}$$

are in $(A+B+C)\mathbb{Z}[A,B,C]$. Define $\mathcal{R}_{2d+1}(A,B,C)$ and $\mathcal{R}_{2d}(A,B,C)$ to be the polynomials in $\mathbb{Z}[A,B,C]$ which satisfy

$$\begin{aligned} \mathfrak{P}'_{2d+1}(A,B,C) + u_{2d+1}A^{2d+1} &= (A+B+C)\mathcal{R}_{2d+1}(A,B,C) \text{ and} \\ \mathfrak{P}'_{2d}(A,B,C) + u_{2d}A^{2d} &= (A+B+C)\mathcal{R}_{2d}(A,B,C). \end{aligned} \tag{3.10}$$

We now focus on assertion (1); that is, we have $\theta = 2d+1$ is odd. Recall, from (2.1), that φ is the matrix $\varphi_{r,n-r}$ of (1.20). It follows that $\varphi\check{}$ is the matrix $-\varphi_{n-r,r}$. We see from (1.21) that $\varphi\varphi\check{} = \varphi\check{}\varphi = fI$. Define ψ to be the matrix

$$\begin{bmatrix} 0 & (-1)^d z^r \operatorname{Poly}'_{d,d+1,d}(z^n,y^n) & y^r \operatorname{Poly}'_{d,d+1,d}(y^n,z^n) & 0 \\ z^r \operatorname{Poly}'_{d,d+1,d}(z^n,x^n) & 0 & (-1)^d x^r \operatorname{Poly}'_{d,d+1,d}(x^n,z^n) & 0 \\ (-1)^d y^r \operatorname{Poly}'_{d,d+1,d}(y^n,x^n) & x^r \operatorname{Poly}'_{d,d+1,d}(x^n,y^n) & 0 & 0 \end{bmatrix}$$

and

$$\Phi = \begin{bmatrix} 0 & z^r\mathcal{R}_{2d+1}(z^n,x^n,y^n) & -y^r\mathcal{R}_{2d+1}(y^n,x^n,z^n) \\ -z^r\mathcal{R}_{2d+1}(z^n,x^n,y^n) & 0 & x^r\mathcal{R}_{2d+1}(x^n,y^n,z^n) \\ y^r\mathcal{R}_{2d+1}(y^n,x^n,z^n) & -x^r\mathcal{R}_{2d+1}(x^n,y^n,z^n) & 0 \end{bmatrix}.$$

One may easily calculate that $\psi\varphi\check{\ }\psi^{\mathrm{T}}$ is equal to

$$\begin{bmatrix} 0 & -z^r\mathfrak{P}'_{2d+1}(z^n,x^n,y^n) & y^r\mathfrak{P}'_{2d+1}(y^n,x^n,z^n) \\ z^r\mathfrak{P}'_{2d+1}(z^n,x^n,y^n) & 0 & -x^r\mathfrak{P}'_{2d-1}(x^n,y^n,z^n) \\ -y^r\mathfrak{P}'_{2d+1}(y^n,x^n,z^n) & x^r\mathfrak{P}'_{2d+1}(x^n,y^n,z^n) & 0 \end{bmatrix}.$$

Therefore, the definition (3.10) of $\mathcal{R}_{2d+1}$ tells us that

$$\psi\varphi\check{\ }\psi^{\mathrm{T}} + f\Phi = u_{2d+1}\boldsymbol{X}.$$

The condition (2.2) is satisfied. Lemma 2.3 applies. One completes the proof of (1) when $\operatorname{char}\boldsymbol{k} = 3$ just like one completes the proof of (1) when $\operatorname{char}\boldsymbol{k} \neq 3$. Once one has the resolution of Q written in terms of the parameter d, then the equation $d = \delta - 1$ may be used to write the resolution in the form which is recorded in (1).

Now that all of the words have been defined, one completes the proof of $\theta = 2d$ and $\operatorname{char}\boldsymbol{k} = 3$ using the exact same words as were used in the proof of $\theta = 2d$ and $\operatorname{char}\boldsymbol{k} \neq 3$. $\square$

Section 4. The proof of Theorem 1.9.

In this section we prove Theorem 1.9: Proposition 4.2 is the second assertion and Proposition 4.9 is the first assertion. The proof is carried out by induction. The key step in this induction is Lemmas 4.1 and 4.8 where we show how the numbers of the form $\binom{A}{D}_{\#p}$ involved in Theorem 1.9, with $A = ap + r$ and $D = dp + \epsilon$, for small values of r and ϵ, are related to numbers of the form $\binom{a}{d}_{\#p}$.

Lemma 4.1. *Let p, d, and ϵ be integers with $p \geq 3$ and prime, $d \geq 1$, and $-\frac{p}{3} < \epsilon < \frac{p}{3}$. Let a, r, s, and t be non-negative integers with $r \leq p-1$, $s \leq 1$, and $t \leq 2$. If $A = pa + r$ and $D = pd + \epsilon$, then*

(a)

$$\binom{2D}{D}_{\#p} = \begin{cases} \left(pd\binom{2d}{d}\right)_{\#p} & \text{if } \epsilon < 0 \\ \binom{2d}{d}_{\#p} & \text{if } 0 \leq \epsilon, \end{cases}$$

(b)

$$\binom{3D}{D}_{\#p} = \begin{cases} \left(\frac{pd}{3}\binom{3d}{d}\right)_{\#p} & \text{if } \epsilon < 0 \\ \binom{3d}{d}_{\#p} & \text{if } 0 \leq \epsilon, \end{cases}$$

(c)

$$\binom{A}{D-s}_{\#p} = \begin{cases} \left(p(a-d)\binom{a}{d}\right)_{\#p} & \text{if } r+s+1-\epsilon \le 0 \\ \binom{a}{d}_{\#p} & \text{if } s \le \epsilon \text{ and } 1 \le r+s+1-\epsilon \le p \\ \left(pd\binom{a}{d}\right)_{\#p} & \text{if } \epsilon \le s-1 \text{ and } 1 \le r+s+1-\epsilon \le p \\ \binom{a}{d-1}_{\#p} & \text{if } \epsilon \le s-1 \text{ and } p+1 \le r+s+1-\epsilon, \end{cases}$$

(d) *if* $0 \le \epsilon$, *then*

$$\binom{3D-A-t}{D}_{\#p} = \begin{cases} \binom{3d-a-2}{d}_{\#p} & \text{if } 3\epsilon+p+1 \le r+t \\ \left(p(2d-a-1)\binom{3d-a-1}{d}\right)_{\#p} & \text{if } 2\epsilon+p+1 \le r+t \le 3\epsilon+p \\ \binom{3d-a-1}{d}_{\#p} & \text{if } 3\epsilon+1 \le r+t \le p+2\epsilon \\ \left(p(2d-a)\binom{3d-a}{d}\right)_{\#p} & \text{if } 2\epsilon+1 \le r+t \le 3\epsilon \\ \binom{3d-a}{d}_{\#p} & \text{if } r+t \le 2\epsilon, \text{ and} \end{cases}$$

(e) *if* $\epsilon \le -1$, *then*

$$\binom{3D-A-t}{D}_{\#p} = \begin{cases} \left(pd\binom{3d-a-2}{d}\right)_{\#p} & \text{if } 2\epsilon+p+1 \le r+t \\ \binom{3d-a-2}{d-1}_{\#p} & \text{if } 3\epsilon+p+1 \le r+t \le 2\epsilon+p \\ \left(pd\binom{3d-a-1}{d}\right)_{\#p} & \text{if } r+t \le p+3\epsilon. \end{cases}$$

Proof. The key to all of these calculations is the observation that if L is a set of integers, then

$$\left(\prod_{\{\ell\in L\}} \ell\right)_{\#p} = \left(\prod_{\{\ell\in L \mid\, p \text{ divides } \ell\}} \ell\right)_{\#p}.$$

In particular, for example, if $0 \le \epsilon < \frac{p}{3}$, then

$$((pd+\epsilon)!)_{\#p} = ([p(d)]\cdot[p(d-1)]\cdots[p(2)]\cdot[p(1)])_{\#p} = \left(p^d d!\right)_{\#p}.$$

Assertion (a) is easy to verify if $d=1$. We assume $2 \le d$. We have

$$\binom{2D}{D}_{\#p} = \left(\frac{(2D)\cdots(D+1)}{D!}\right)_{\#p} = \left(\frac{[p(2d)+2\epsilon]\cdots[p(d)+\epsilon+1]}{(pd+\epsilon)!}\right)_{\#p}$$

$$= \begin{cases} \left(\frac{[p(2d-1)]\cdots[pd]}{[p(d-1)]\cdots[p(1)]}\right)_{\#p} = \left(pd\binom{2d-1}{d-1}\right)_{\#p} & \text{if } \epsilon < 0 \\ \left(\frac{[p(2d)]\cdots[p(d+1)]}{[p(d)]\cdots[p(1)]}\right)_{\#p} = \binom{2d}{d}_{\#p} & \text{if } 0 \le \epsilon. \end{cases}$$

Notice that $\binom{2d-1}{d-1} = \frac{1}{2}\binom{2d}{d}$ and $\left(\frac{1}{2}\binom{2d}{d}\right)_{\#p} = \binom{2d}{d}_{\#p}$, since $p \neq 2$.

The technique that was used for (a) shows that

$$\binom{3D}{D}_{\#p} = \begin{cases} \left(pd\binom{3d-1}{d-1}\right)_{\#p} & \text{if } \epsilon < 0 \\ \binom{3d}{d}_{\#p} & \text{if } 0 \le \epsilon. \end{cases}$$

Furthermore, we know that $\binom{3d-1}{d-1} = \frac{1}{3}\binom{3d}{d}$.

We now prove (c). We know that $\binom{A}{D-s}_{\#p}$ is equal to

$$\left(\frac{A\cdots(A-D+s+1)}{(D-s)!}\right)_{\#p} = \left(\frac{[pa+r]\cdots[p(a-d)+r+s+1-\epsilon]}{(pd+\epsilon-s)!}\right)_{\#p}.$$

Notice that that $-p < \epsilon - s < p$ and that $-p < r+s+1-\epsilon < 2p$. It follows that

$$((pd+\epsilon-s)!)_{\#p} = \begin{cases} ([p(d)]\cdots[p(1)])_{\#p} & \text{if } s \le \epsilon \\ ([p(d-1)]\cdots[p(1)])_{\#p} & \text{if } \epsilon \le s-1 \end{cases}$$

and $([pa+r]\cdots[p(a-d)+r+s+1-\epsilon])_{\#p}$

$$= \begin{cases} ([pa]\cdots[p(a-d)])_{\#p} & \text{if } r+s+1-\epsilon \le 0 \\ ([pa]\cdots[p(a-d+1)])_{\#p} & \text{if } 1 \le r+s+1-\epsilon \le p \\ ([pa]\cdots[p(a-d+2)])_{\#p} & \text{if } p+1 \le r+s+1-\epsilon. \end{cases}$$

Observe that $s \le \epsilon \implies r+s+1-\epsilon \le p$ and $\epsilon \le s-1 \implies 1 \le r+s+1-\epsilon$. There are a total of four options: $\binom{A}{D-s}_{\#p}$ is equal to

$$\begin{cases} \left(\frac{[pa]\cdots[p(a-d)]}{[p(d)]\cdots[p(1)]}\right)_{\#p} = \left(p(a-d)\binom{a}{d}\right)_{\#p} & \text{if } s \le \epsilon \text{ and } r+s+1-\epsilon \le 0 \\ \left(\frac{[pa]\cdots[p(a-d+1)]}{[p(d)]\cdots[p(1)]}\right)_{\#p} = \binom{a}{d}_{\#p} & \text{if } s \le \epsilon \text{ and } 1 \le r+s+1-\epsilon \le p \\ \left(\frac{[pa]\cdots[p(a-d+1)]}{[p(d-1)]\cdots[p(1)]}\right)_{\#p} = \left(pd\binom{a}{d}\right)_{\#p} & \text{if } \epsilon \le s-1 \text{ and } 1 \le r+s+1-\epsilon \le p \\ \left(\frac{[pa]\cdots[p(a-d+2)]}{[p(d-1)]\cdots[p(1)]}\right)_{\#p} = \binom{a}{d-1}_{\#p} & \text{if } \epsilon \le s-1 \text{ and } p+1 \le r+s+1-\epsilon. \end{cases}$$

In the top case, $r+s+1-\epsilon \le 0 \implies 1 \le r+1 \le \epsilon - s$, so the constraint $s \le \epsilon$ is redundant. This calculation is valid for $1 \le d$; however, special care is needed when $d = 1$. In this case, p does not divide $(pd+\epsilon-s)!$ when $\epsilon - s \le -1$ and p does not divide $[pa+r]\cdots[p(a-d)+r+s+1-\epsilon]$ when $p+1 \le r+s+1-\epsilon$.

We now prove (d) and (e). The argument we give is valid for all $d \ge 1$. We have

$$\binom{3D-A-t}{D}_{\#p} = \left(\frac{(3D-A-t)\cdots(2D-A-t+1)}{D!}\right)_{\#p}$$
$$= \left(\frac{[p(3d-a)+3\epsilon-r-t]\cdots[p(2d-a)+2\epsilon-r-t+1]}{(pd+\epsilon)!}\right)_{\#p}.$$

We know that

$$((pd+\epsilon)!)_{\#p} = \begin{cases} ([p(d)]\cdots[p(1)])_{\#p} & \text{if } 0 \le \epsilon \\ \left(p^{d-1}(d-1)!\right)_{\#p} & \text{if } \epsilon \le -1. \end{cases}$$

We see that

$$0 \le \epsilon \implies -2p < 3\epsilon - r - t < p \quad \text{and} \quad \epsilon \le -1 \implies -2p \le 3\epsilon - r - t < 0.$$

Thus, $\binom{3D-A-t}{D}_{\#p}$ is equal to

$$\begin{cases} \left(\frac{[p(3d-a-2)]\cdots[p(2d-a)+2\epsilon-r-t+1]}{[p(d)]\cdots[p(1)]}\right)_{\#p} & \text{if } 0 \le \epsilon \text{ and } 3\epsilon-r-t \le -p-1 \\ \left(\frac{[p(3d-a-1)]\cdots[p(2d-a)+2\epsilon-r-t+1]}{[p(d)]\cdots[p(1)]}\right)_{\#p} & \text{if } 0 \le \epsilon \text{ and } -p \le 3\epsilon-r-t \le -1 \\ \left(\frac{[p(3d-a)]\cdots[p(2d-a)+2\epsilon-r-t+1]}{[p(d)]\cdots[p(1)]}\right)_{\#p} & \text{if } 0 \le \epsilon \text{ and } 0 \le 3\epsilon-r-t \\ \left(\frac{[p(3d-a-2)]\cdots[p(2d-a)+2\epsilon-r-t+1]}{p^{d-1}(d-1)!}\right)_{\#p} & \text{if } \epsilon \le -1 \text{ and } 3\epsilon-r-t \le -p-1 \\ \left(\frac{[p(3d-a-1)]\cdots[p(2d-a)+2\epsilon-r-t+1]}{p^{d-1}(d-1)!}\right)_{\#p} & \text{if } \epsilon \le -1 \text{ and } -p \le 3\epsilon-r-t. \end{cases}$$

We see that

$$-2p < -\frac{5p}{3} = -\frac{2p}{3} - (p-1) - 2 + 1 \le 2\epsilon - r - t + 1 < \frac{2p}{3} + 1 \le p;$$

and therefore, the smallest multiple of p which is at least $[p(2d-a)+2\epsilon-r-t+1]$ is

$$\begin{cases} p(2d-a-1) & \text{if } 2\epsilon-r-t+1 \le -p \\ p(2d-a) & \text{if } -p+1 \le 2\epsilon-r-t+1 \le 0 \\ p(2d-a+1) & \text{if } 1 \le 2\epsilon-r-t+1. \end{cases}$$

If $0 \le \epsilon$, then $2\epsilon \le 3\epsilon \le p+2\epsilon \le p+3\epsilon$ and the constraints

$$\begin{cases} 3\epsilon - r - t \le -p-1 \\ -p \le 3\epsilon - r - t \le -1 \\ 0 \le 3\epsilon - r - t \end{cases} \quad \text{and} \quad \begin{cases} 2\epsilon - r - t + 1 \le -p \\ -p+1 \le 2\epsilon - r - t + 1 \le 0 \\ 1 \le 2\epsilon - r - t + 1 \end{cases}$$

may be merged to produce

$$\begin{cases} 3\epsilon + p + 1 \le r+t \\ 2\epsilon + p + 1 \le r+t \le 3\epsilon + p \\ 3\epsilon + 1 \le r+t \le p+2\epsilon \\ 2\epsilon + 1 \le r+t \le 3\epsilon \\ r+t \le 2\epsilon. \end{cases}$$

Thus, if $0 \le \epsilon$, then $\binom{3D-A-t}{D}_{\#p}$ is equal to

$$\begin{cases} \left(\frac{[p(3d-a-2)]\cdots[p(2d-a-1)]}{[p(d)]\cdots[p(1)]}\right)_{\#p} = \binom{3d-a-2}{d}_{\#p} & \text{if } 3\epsilon+p+1 \le r+t \\ \left(\frac{[p(3d-a-1)]\cdots[p(2d-a-1)]}{[p(d)]\cdots[p(1)]}\right)_{\#p} = \left(p(2d-a-1)\binom{3d-a-1}{d}\right)_{\#p} & \text{if } 2\epsilon+p+1 \le r+t \le 3\epsilon+p \\ \left(\frac{[p(3d-a-1)]\cdots[p(2d-a)]}{[p(d)]\cdots[p(1)]}\right)_{\#p} = \binom{3d-a-1}{d}_{\#p} & \text{if } 3\epsilon+1 \le r+t \le p+2\epsilon \\ \left(\frac{[p(3d-a)]\cdots[p(2d-a)]}{[p(d)]\cdots[p(1)]}\right)_{\#p} = \left(p(2d-a)\binom{3d-a}{d}\right)_{\#p} & \text{if } 2\epsilon+1 \le r+t \le 3\epsilon \\ \left(\frac{[p(3d-a)]\cdots[p(2d-a+1)]}{[p(d)]\cdots[p(1)]}\right)_{\#p} = \binom{3d-a}{d}_{\#p} & \text{if } r+t \le 2\epsilon. \end{cases}$$

In a similar manner, if $\epsilon \le -1$, then

$$2\epsilon < 0 < 3\epsilon + p < 2\epsilon + p$$

and the constraints

$$\begin{cases} 3\epsilon - r - t \le -p-1 \\ -p \le 3\epsilon - r - t \end{cases} \quad \text{and} \quad \begin{cases} 2\epsilon - r - t + 1 \le -p \\ -p+1 \le 2\epsilon - r - t + 1 \le 0 \\ 1 \le 2\epsilon - r - t + 1 \end{cases}$$

may be merged to produce

$$\begin{cases} 2\epsilon + p + 1 \le r+t \\ 3\epsilon + p + 1 \le r+t \le 2\epsilon + p \\ r+t \le p+3\epsilon, \end{cases}$$

since $r+t$ is automatically non-negative. Therefore, if $\epsilon \le -1$, then $\binom{3D-A-t}{D}_{\#p}$ is equal to

$$\begin{cases} \left(\frac{[p(3d-a-2)]\cdots[p(2d-a-1)]}{p^{d-1}(d-1)!}\right)_{\#p} = \left(pd\binom{3d-a-2}{d}\right)_{\#p} & \text{if } 2\epsilon+p+1 \le r+t \\ \left(\frac{[p(3d-a-2)]\cdots[p(2d-a)]}{p^{d-1}(d-1)!}\right)_{\#p} = \binom{3d-a-2}{d-1}_{\#p} & \text{if } 3\epsilon+p+1 \le r+t \le 2\epsilon+p \\ \left(\frac{[p(3d-a-1)]\cdots[p(2d-a)]}{p^{d-1}(d-1)!}\right)_{\#p} = \left(pd\binom{3d-a-1}{d}\right)_{\#p} & \text{if } r+t \le p+3\epsilon. \quad \square \end{cases}$$

Proposition 4.2. *Assertion* (2) *of Theorem* 1.9 *holds.*

Proof. Fix a prime integer p with $p \ge 5$. All four assertions are clear for $0 \le d < \frac{p}{3}$ because p does not divide either $\binom{2d}{d}$ or $\binom{3d}{d}$. The proof proceeds by induction. Let D be an element of D_p which is greater than $\frac{p}{3}$. The recursive definition of D_p guarantees that $D = pd + \epsilon$ for some $d \in D_p$ and some integer ϵ with $-\frac{p}{3} < \epsilon < \frac{p}{3}$. We see that $d < D$; so the induction hypothesis guarantees that $\binom{2d}{d}_{\#p} = \binom{3d}{d}_{\#p}$. According to Lemma 4.1 we have

$$\begin{cases} \binom{2D}{D}_{\#p} = \binom{2d}{d}_{\#p} = \binom{3d}{d}_{\#p} = \binom{3D}{D}_{\#p} & \text{if } 0 \le \epsilon \text{ and} \\ \binom{2D}{D}_{\#p} = \left(pd\binom{2d}{d}\right)_{\#p} = \left(pd\binom{3d}{d}\right)_{\#p} = \left(\frac{pd}{3}\binom{3d}{d}\right)_{\#p} = \binom{3D}{D}_{\#p} & \text{if } \epsilon \le -1. \end{cases}$$

So $\binom{2D}{D}_{\#p} = \binom{3D}{D}_{\#p}$ in all cases and assertion (a) is established.

The induction hypothesis also guarantees that all of the inequalities (b), (c), and (d) hold when $0 \le a \le 2d$. Notice also that the induction hypothesis ensures that

$$\left(\binom{3d-a-2}{d-1}\binom{a}{d}\right)_{\#p} \ge \binom{2d}{d}_{\#p} \tag{4.3}$$

also holds for $0 \le a \le 2d$ because this is a different form of (d).

Fix $A = pa + r$ with

$$0 \le a, \quad 0 \le r \le p-1, \quad \text{and} \quad 0 \le A \le 2D.$$

We prove that all three inequalities (b)–(d) hold at the pair (A, D).

Observe first that $0 \le a \le 2d$; so all four inequalities (b)–(d) and (4.3) hold at (a, d) by induction.

We first establish (b) and (c) at (A, D), when $0 \le \epsilon$. Fix t with $0 \le t \le 1$ and $0 \le \epsilon$. We study

$$\left(\binom{A}{D}\binom{3D-A-t}{A}\right)_{\#p}.$$

Lemma 4.1 shows that

$$\binom{A}{D}_{\#p} = \begin{cases} \left(p(a-d)\binom{a}{d}\right)_{\#p} & \text{if } r \le \epsilon - 1 \\ \binom{a}{d}_{\#p} & \text{if } \epsilon \le r. \end{cases} \tag{4.4}$$

The hypotheses $0 \le \epsilon$ and $t \le 1$ ensure that $r+t \le 3\epsilon + p$; so, Lemma 4.1 also gives

$$\binom{3D-A-t}{A}_{\#p} = \begin{cases} \left(p(2d-a-1)\binom{3d-a-1}{d}\right)_{\#p} & \text{if } 2\epsilon + p + 1 \le r+t \\ \binom{3d-a-1}{d}_{\#p} & \text{if } 3\epsilon + 1 \le r+t \le p + 2\epsilon \\ \left(p(2d-a)\binom{3d-a}{d}\right)_{\#p} & \text{if } 2\epsilon + 1 \le r+t \le 3\epsilon \\ \binom{3d-a}{d}_{\#p} & \text{if } r+t \le 2\epsilon. \end{cases} \tag{4.5}$$

We combine (4.4) and (4.5). Observe that the condition $r \le \epsilon - 1$ forces $r+t \le 2\epsilon$. It follows that $\left(\binom{A}{D}\binom{3D-A-t}{A}\right)_{\#p}$ is equal to

$$\begin{cases} \left(\binom{a}{d}p(2d-a-1)\binom{3d-a-1}{d}\right)_{\#p} & \text{if } 2\epsilon + p + 1 \le r+t \\ \left(\binom{a}{d}\binom{3d-a-1}{d}\right)_{\#p} & \text{if } 3\epsilon + 1 \le r+t \le p + 2\epsilon \\ \left(\binom{a}{d}p(2d-a)\binom{3d-a}{d}\right)_{\#p} & \text{if } 2\epsilon + 1 \le r+t \le 3\epsilon \\ \left(\binom{a}{d}\binom{3d-a}{d}\right)_{\#p} & \text{if } r+t \le 2\epsilon \text{ and } \epsilon \le r \\ \left(p(a-d)\binom{a}{d}\binom{3d-a}{d}\right)_{\#p} & \text{if } r+t \le 2\epsilon \text{ and } r \le \epsilon - 1. \end{cases}$$

The hypothesis that all (b)–(d) hold at (a,d) guarantees that

$$\left(\binom{A}{D}\binom{3D-A-t}{A}\right)_{\#p} \ge \binom{2d}{d}_{\#p} = \binom{2D}{D}_{\#p},$$

as claimed.

We next establish (b) and (c) at (A,D) simultaneously, when $\epsilon \le -1$. Fix t with $0 \le t \le 1$ and $\epsilon \le -1$. We study

$$\left(\binom{A}{D}\binom{3D-A-t}{A}\right)_{\#p}.$$

Apply part (c) of Lemma 4.1 with $s = 0$ and $\epsilon \le -1$. The cases $r+s+1-\epsilon \le 0$ and $s \le \epsilon$ do not occur in this situation and the condition $\epsilon \le r$ holds automatically; so, we have

$$\binom{A}{D}_{\#p} = \begin{cases} \left(pd\binom{a}{d}\right)_{\#p} & r \le p-1+\epsilon \\ \binom{a}{d-1}_{\#p} & p+\epsilon \le r. \end{cases} \tag{4.6}$$

Part (e) of Lemma 4.1 gives

$$\binom{3D-A-t}{D}_{\#p} = \begin{cases} \left(pd\binom{3d-a-2}{d}\right)_{\#p} & \text{if } 2\epsilon+p+1 \le r+t \\ \binom{3d-a-2}{d-1}_{\#p} & \text{if } 3\epsilon+p+1 \le r+t \le 2\epsilon+p \\ \left(pd\binom{3d-a-1}{d}\right)_{\#p} & \text{if } r+t \le p+3\epsilon. \end{cases} \tag{4.7}$$

We notice that $p+\epsilon \le r$ implies

$$p+2\epsilon+1 = p+\epsilon+(\epsilon+1) \le r+0 \le r+t;$$

furthermore, $p+3\epsilon < p+2\epsilon < p+2\epsilon+1$. Combine (4.6) and (4.7) to see that $\left(\binom{A}{D}\binom{3D-A-t}{D}\right)_{\#p}$ is equal to

$$\begin{cases} \left(\binom{a}{d-1}pd\binom{3d-a-2}{d}\right)_{\#p} & \text{if } 2\epsilon+p+1 \le r+t \text{ and } p+\epsilon \le r \\ \left(pd\binom{a}{d}pd\binom{3d-a-2}{d}\right)_{\#p} & \text{if } 2\epsilon+p+1 \le r+t \text{ and } r \le p-1+\epsilon \\ \left(pd\binom{a}{d}\binom{3d-a-2}{d-1}\right)_{\#p} & \text{if } 3\epsilon+p+1 \le r+t \le 2\epsilon+p \\ \left(pd\binom{a}{d}pd\binom{3d-a-1}{d}\right)_{\#p} & \text{if } r+t \le p+3\epsilon. \end{cases}$$

Notice that $d\binom{a}{d} = \binom{a}{d-1}(a-d+1)$; so the second option in the above display is $\left(p(a-d+1)\binom{a}{d-1}pd\binom{3d-a-2}{d}\right)_{\#p}$. For each case, apply one of the inequalities (b)–(d) or (4.3) at (a,d). We conclude that

$$\left(\binom{A}{D}\binom{3D-A-t}{D}\right)_{\#p} \ge \left(pd\binom{2d}{d}\right)_{\#p} = \binom{2D}{D}_{\#p}.$$

We next establish (d) when $0 \le \epsilon$. Lemma 4.1 gives

$$\binom{A}{D-1}_{\#p} = \begin{cases} \left(p(a-d)\binom{a}{d}\right)_{\#p} & \text{if } r \le \epsilon - 2 \\ \binom{a}{d}_{\#p} & \text{if } 1 \le \epsilon \text{ and } \epsilon - 1 \le r \le p-2+\epsilon \\ \left(pd\binom{a}{d}\right)_{\#p} & \text{if } \epsilon = 0 \text{ and } r \le p-2 \\ \binom{a}{d-1}_{\#p} & \text{if } \epsilon = 0 \text{ and } p-1 = r \end{cases}$$

and $\binom{3D-A-2}{D}_{\#p}$ is equal to

$$\begin{cases} \binom{3d-a-2}{d}_{\#p} & \text{if } 3\epsilon + p - 1 \le r \\ \left(p(2d-a-1)\binom{3d-a-1}{d}\right)_{\#p} & \text{if } 2\epsilon + p - 1 \le r \le 3\epsilon + p - 2 \\ \binom{3d-a-1}{d}_{\#p} & \text{if } 3\epsilon - 1 \le r \le p - 2 + 2\epsilon \\ \left(p(2d-a)\binom{3d-a}{d}\right)_{\#p} & \text{if } 2\epsilon - 1 \le r \le 3\epsilon - 2 \\ \binom{3d-a}{d}_{\#p} & \text{if } r \le 2\epsilon - 2. \end{cases}$$

First consider $\epsilon = 0$. In this case,

$$\binom{A}{D-1}_{\#p} = \begin{cases} \left(pd\binom{a}{d}\right)_{\#p} & \text{if } r \le p-2 \\ \binom{a}{d-1}_{\#p} & \text{if } p-1 = r \end{cases}$$

and

$$\binom{3D-A-2}{D}_{\#p} = \begin{cases} \binom{3d-a-2}{d}_{\#p} & \text{if } p-1 \le r \\ \binom{3d-a-1}{d}_{\#p} & \text{if } r \le p-2; \end{cases}$$

so,

$$\left(\binom{A}{D-1}_{\#p}\binom{3D-A-2}{D}\right)_{\#p} = \begin{cases} \binom{a}{d-1}_{\#p}\binom{3d-a-2}{d}_{\#p} & \text{if } p-1 \le r \\ \left(pd\binom{a}{d}\right)_{\#p}\binom{3d-a-1}{d}_{\#p} & \text{if } r \le p-2, \end{cases}$$

which is at least $\binom{2d}{d}_{\#p} = \binom{2D}{D}_{\#p}$. Now consider $1 \le \epsilon$. We have

$$\binom{A}{D-1}_{\#p} = \begin{cases} \left(p(a-d)\binom{a}{d}\right)_{\#p} & \text{if } r \le \epsilon - 2 \\ \binom{a}{d}_{\#p} & \text{if } \epsilon - 1 \le r \end{cases}$$

and (since $3\epsilon+p-1 \le r \le p-1$ is impossible) $\binom{3D-A-2}{D}_{\#p}$ is equal to

$$\begin{cases} \left(p(2d-a-1)\binom{3d-a-1}{d}\right)_{\#p} & \text{if } 2\epsilon+p-1 \le r \le 3\epsilon+p-2 \\ \binom{3d-a-1}{d}_{\#p} & \text{if } 3\epsilon-1 \le r \le p-2+2\epsilon \\ \left(p(2d-a)\binom{3d-a}{d}\right)_{\#p} & \text{if } 2\epsilon-1 \le r \le 3\epsilon-2 \\ \binom{3d-a}{d}_{\#p} & \text{if } r \le 2\epsilon-2. \end{cases}$$

Of course, if $r \le \epsilon-2$, then $r \le 2\epsilon-2$; so, $\left(\binom{A}{D-1}\binom{3D-A-2}{D}\right)_{\#p}$ is equal to

$$\begin{cases} \left(\binom{a}{d}p(2d-a-1)\binom{3d-a-1}{d}\right)_{\#p} & \text{if } 2\epsilon+p-1 \le r \le 3\epsilon+p-2 \\ \left(\binom{a}{d}\binom{3d-a-1}{d}\right)_{\#p} & \text{if } 3\epsilon-1 \le r \le p-2+2\epsilon \\ \left(\binom{a}{d}p(2d-a)\binom{3d-a}{d}\right)_{\#p} & \text{if } 2\epsilon-1 \le r \le 3\epsilon-2 \\ \left(\binom{a}{d}\binom{3d-a}{d}\right)_{\#p} & \text{if } \epsilon-1 \le r \le 2\epsilon-2 \\ \left(p(a-d)\binom{a}{d}\binom{3d-a}{d}\right)_{\#p} & \text{if } r \le \epsilon-2, \end{cases}$$

which is at least $\binom{2d}{d}_{\#p} = \binom{2D}{D}_{\#p}$.

Finally, we study (d) with $\epsilon \le -1$. We have

$$\binom{A}{D-1}_{\#p} = \begin{cases} \left(pd\binom{a}{d}\right)_{\#p} & \text{if } r \le p-2+\epsilon \\ \binom{a}{d-1}_{\#p} & \text{if } p-1+\epsilon \le r \end{cases}$$

and

$$\binom{3D-A-2}{D}_{\#p} = \begin{cases} \left(pd\binom{3d-a-2}{d}\right)_{\#p} & \text{if } 2\epsilon+p-1 \le r \\ \binom{3d-a-2}{d-1}_{\#p} & \text{if } 3\epsilon+p-1 \le r \le 2\epsilon+p-2 \\ \left(pd\binom{3d-a-1}{d}\right)_{\#p} & \text{if } r \le p+3\epsilon-2. \end{cases}$$

Of course, $p-1+2\epsilon < p-1+\epsilon$ (because $\epsilon < 0$). Thus, $\left(\binom{A}{D-1}\binom{3D-A-2}{D}\right)_{\#p}$ is equal to

$$\begin{cases} \left(\binom{a}{d-1}pd\binom{3d-a-2}{d}\right)_{\#p} & p-1+\epsilon \le r \\ \left(pd\binom{a}{d}pd\binom{3d-a-2}{d}\right)_{\#p} & \text{if } 2\epsilon+p-1 \le r \le p-2+\epsilon \\ \left(pd\binom{a}{d}\binom{3d-a-2}{d-1}\right)_{\#p} & \text{if } 3\epsilon+p-1 \le r \le 2\epsilon+p-2 \\ \left(pd\binom{a}{d}pd\binom{3d-a-1}{d}\right)_{\#p} & \text{if } r \le p+3\epsilon-2. \end{cases}$$

In the second line, we use $d\binom{a}{d} = \binom{a}{d-1}(a-d+1)$. In the third line we use (4.3). We conclude that

$$\left(\binom{A}{D-1}\binom{3D-A-2}{D}\right)_{\#p} \geq \left(pd\binom{2d}{d}\right)_{\#p} = \binom{2D}{D}_{\#p}. \quad \square$$

Lemma 4.8. *If $A = 3a + r$ with $0 \leq r \leq 2$ and $D = 3d + \epsilon$ with ϵ equal to 0 or 2, then*

(a) $\binom{A}{D}_{\#3} = \begin{cases} \binom{a}{d}_{\#3} & \text{if } \epsilon = 0 \text{ or } \epsilon = r = 2 \\ \left(3(d+1)\binom{a}{d+1}\right)_{\#3} & \text{if } \epsilon = 2 \text{ and } 0 \leq r \leq 1, \end{cases}$

(b) $\binom{2D}{D}_{\#3} = \begin{cases} \binom{2d}{d}_{\#3} & \text{if } \epsilon = 0 \\ \left(3(d+1)\binom{2d+1}{d}\right)_{\#3} & \text{if } \epsilon = 2, \end{cases}$

(c) $\binom{2D+1}{D}_{\#3} = \begin{cases} \binom{2d}{d}_{\#3} & \text{if } \epsilon = 0 \\ \binom{2d+1}{d}_{\#3} & \text{if } \epsilon = 2, \end{cases}$

(d) $\binom{3D}{D}_{\#3} = \begin{cases} \binom{3d}{d}_{\#3} & \text{if } \epsilon = 0 \\ \left(3(d+1)\binom{3d+2}{d}\right)_{\#3} & \text{if } \epsilon = 2, \end{cases}$

(e) $\binom{3D+2}{D}_{\#3} = \begin{cases} \binom{3d}{d}_{\#3} & \text{if } \epsilon = 0 \\ \binom{3d+2}{d}_{\#3} & \text{if } \epsilon = 2, \end{cases}$

(f) $\binom{3D-1-A}{D}_{\#3} = \begin{cases} \binom{3d-1-a}{d}_{\#3} & \text{if } \epsilon = 0 \\ \binom{3d+1-a}{d}_{\#3} & \text{if } \epsilon = 2 \text{ and } r = 0 \\ \left(3(2d-a+1)\binom{3d+1-a}{d}\right)_{\#3} & \text{if } \epsilon = 2 \text{ and } 1 \leq r \leq 2, \end{cases}$

(g) $\binom{3D-A}{D}_{\#3} = \begin{cases} \binom{3d-a}{d}_{\#3} & \text{if } \epsilon = r = 0 \\ \binom{3d-1-a}{d}_{\#3} & \text{if } \epsilon = 0 \text{ and } 1 \leq r \leq 2 \\ \left(3(3d-a+2)\binom{3d+1-a}{d}\right)_{\#3} & \text{if } \epsilon = 2 \text{ and } r = 0 \\ \binom{3d+1-a}{d}_{\#3} & \text{if } \epsilon = 2 \text{ and } r = 1 \\ \left(3(d+1)\binom{3d+1-a}{d+1}\right)_{\#3} & \text{if } \epsilon = r = 2, \text{ and} \end{cases}$

(h) $\binom{3D+1-A}{D+1}_{\#3} = \begin{cases} \binom{3d-a}{d}_{\#3} & \textit{if } \epsilon = r = 0 \\ \left(3(2d-a)\binom{3d-a}{d}\right)_{\#3} & \textit{if } \epsilon = 0 \textit{ and } r = 1 \\ \binom{3d-1-a}{d}_{\#3} & \textit{if } \epsilon = 0 \textit{ and } r = 2 \\ \binom{3d+2-a}{d+1}_{\#3} & \textit{if } \epsilon = 2 \textit{ and } 0 \le r \le 1 \\ \binom{3d+1-a}{d+1}_{\#3} & \textit{if } \epsilon = r = 2. \end{cases}$

Proof. We compute

$$\binom{A}{D}_{\#3} = \left(\frac{A\cdots(A-D+1)}{D!}\right)_{\#3} = \left(\frac{[3a+r]\cdots[3(a-d)+(r+1-\epsilon)]}{(3d+\epsilon)!}\right)_{\#3}$$

$$= \begin{cases} \left(\frac{[3a]\cdots[3(a-d)]}{[3d]\cdots[3(1)]}\right)_{\#3} = \left(3(d+1)\binom{a}{d+1}\right)_{\#3} & \text{if } r+1 \le \epsilon \\ \left(\frac{[3a]\cdots[3(a-d+1)]}{[3d]\cdots[3(1)]}\right)_{\#3} = \binom{a}{d}_{\#3} & \text{if } \epsilon \le r, \end{cases}$$

and (a) is established. One may obtain (b) – (g) from (a) or by direct calculation. We prove (h). Observe that

$$\binom{3D+1-A}{D+1}_{\#3} = \left(\frac{[3(3d+\epsilon-a)+1-r]\cdots[3(2d-a)+(2\epsilon-r+1)]}{(3d+\epsilon+1)!}\right)_{\#3}.$$

We see that the largest multiple of 3 in the numerator is

$$\begin{cases} 3(3d+\epsilon-a-1) & \text{if } r=2 \\ 3(3d+\epsilon-a) & \text{if } 0 \le r \le 1. \end{cases}$$

We see that $-1 \le 2\epsilon - r + 1 \le 5$ and the smallest multiple of 3 in the numerator is

$$\begin{cases} 3(2d-a) & \text{if } 2\epsilon - r + 1 \le 0 \\ 3(2d-a+1) & \text{if } 1 \le 2\epsilon - r + 1 \le 3 \\ 3(2d-a+2) & \text{if } 4 \le 2\epsilon - r + 1. \end{cases}$$

The largest multiple of 3 in the denominator is

$$\begin{cases} 3(d+1) & \text{if } \epsilon = 2 \\ 3d & \text{if } \epsilon = 0. \end{cases}$$

It is not difficult to put the pieces together. $\square$

Proposition 4.9. *Assertion* (1) *of Theorem* 1.9 *holds.*

Proof. It is clear that all five statements hold for $d = 0$. The proof proceeds by induction. Let $D > 0$ be an element of D_3. So $D = 3d + \epsilon$ with $d \in D_3$, $d < D$, and ϵ equal to 0 or 2. The induction hypothesis ensures that all five statements hold for d. Lemma 4.8, together with the induction hypothesis, yields

$$\binom{2D}{D}_{\#3} = \begin{Bmatrix} \binom{2d}{d}_{\#3} & \text{if } \epsilon = 0 \\ \left(3(d+1)\binom{2d+1}{d}\right)_{\#3} & \text{if } \epsilon = 2 \end{Bmatrix} = \begin{Bmatrix} \binom{3d}{d}_{\#3} & \text{if } \epsilon = 0 \\ \left(3(d+1)\binom{3d+2}{d}\right)_{\#3} & \text{if } \epsilon = 2 \end{Bmatrix} = \binom{3D}{D}_{\#3}$$

and

$$\binom{2D+1}{D}_{\#3} = \begin{Bmatrix} \binom{2d}{d}_{\#3} & \text{if } \epsilon = 0 \\ \binom{2d+1}{d}_{\#3} & \text{if } \epsilon = 2 \end{Bmatrix} = \begin{Bmatrix} \binom{3d}{d}_{\#3} & \text{if } \epsilon = 0 \\ \binom{3d+2}{d}_{\#3} & \text{if } \epsilon = 2 \end{Bmatrix} = \binom{3D+2}{D}_{\#3}.$$

Statements (a) and (b) hold at D. For (c), use Lemma 4.8 and the induction hypothesis to see that

$$\left(\binom{A}{D}\binom{3D-A}{D}\right)_{\#3} \geq \begin{cases} \left(\binom{a}{d}\binom{3d-a}{d}\right)_{\#3} & \text{if } \epsilon = r = 0 \\ \left(\binom{a}{d}\binom{3d-1-a}{d}\right)_{\#3} & \text{if } \epsilon = 0 \text{ and } 1 \leq r \\ \left(3(d+1)\binom{a}{d+1}\binom{3d+1-a}{d}\right)_{\#3} & \text{if } \epsilon = 2 \text{ and } r \leq 1 \\ \left(3(d+1)\binom{a}{d}\binom{3d+1-a}{d+1}\right)_{\#3} & \text{if } \epsilon = r = 2 \end{cases}$$

$$\geq \begin{Bmatrix} \binom{2d}{d}_{\#3} & \text{if } \epsilon = 0 \\ \left(3(d+1)\binom{2d+1}{d}\right)_{\#3} & \text{if } \epsilon = 2 \end{Bmatrix} = \binom{2D}{D}_{\#3}.$$

Use Lemma 4.8, the fact that

$$(2d-a+1)\binom{3d+1-a}{d} = (d+1)\binom{3d+1-a}{d+1},$$

and the induction hypothesis to see that

$$\left(\binom{A}{D}\binom{3D-1-A}{D}\right)_{\#3} \geq \begin{cases} \left(\binom{a}{d}\binom{3d-1-a}{d}\right)_{\#3} & \text{if } \epsilon = 0 \\ \left(3(d+1)\binom{a}{d+1}\binom{3d+1-a}{d}\right)_{\#3} & \text{if } \epsilon = 2 \text{ and } r \leq 1 \\ \left(\binom{a}{d}3(d+1)\binom{3d+1-a}{d+1}\right)_{\#3} & \text{if } \epsilon = r = 2 \end{cases}$$

$$\geq \begin{Bmatrix} \binom{2d}{d}_{\#3} & \text{if } \epsilon = 0 \\ \left(3(d+1)\binom{2d+1}{d}\right)_{\#3} & \text{if } \epsilon = 2 \end{Bmatrix} = \binom{2D}{D}_{\#3};$$

thereby verifying (d) at D. Finally, we consider (e). In addition to Lemma 4.8 and the induction hypothesis, we use

$$(d+1)\binom{a}{d+1} = (a-d)\binom{a}{d} \quad \text{and} \quad (d+1)\binom{3d+2-a}{d+1} = (3d+2-a)\binom{3d+1-a}{d},$$

to see that

$$\left(\binom{A}{D}\binom{3D+1-A}{D+1}\right)_{\#3} \geq \begin{cases} \left(\binom{a}{d}\binom{3d-a}{d}\right)_{\#3} & \text{if } \epsilon = 0 \text{ and } r \leq 1 \\ \left(3(d+1)\binom{a}{d}\binom{3d-1-a}{d}\right)_{\#3} & \text{if } \epsilon = 0 \text{ and } r = 2 \\ \left(\binom{a}{d+1}\binom{3d+1-a}{d}\right)_{\#3} & \text{if } \epsilon = 2 \text{ and } r \leq 1 \\ \left(\binom{a}{d}\binom{3d+1-a}{d+1}\right)_{\#3} & \text{if } \epsilon = r = 2 \end{cases}$$

$$\geq \left\{ \begin{array}{ll} \binom{2d}{d}_{\#3} & \text{if } \epsilon = 0 \\ \binom{2d+1}{d}_{\#3} & \text{if } \epsilon = 2 \end{array} \right\} = \binom{2D+1}{D}_{\#3}. \quad \square$$

SECTION 5. THE PROJECTIVE DIMENSION OF $Q_{\boldsymbol{k},n,N}$ IS FINITE WHEN $\lfloor \frac{N}{n} \rfloor$ IS IN T_p.

Throughout this section, $\boldsymbol{k}$ is a field, n is a positive integer, R is the diagonal hypersurface ring $R_{\boldsymbol{k},n}$ of (0.1), and $B = \boldsymbol{k}[X,Y]$ is a polynomial ring.

Notation 5.1. For each positive integer a, we let HB_a denote a homogeneous Hilbert-Burch matrix for the row vector $[\,X^a \quad Y^a \quad (X+Y)^a\,]$ in $B = \boldsymbol{k}[X,Y]$. In particular,

$$0 \to B(-a-d_1) \oplus B(-a-d_2) \xrightarrow{\mathrm{HB}_a} B(-a)^3 \xrightarrow{[\,X^a \quad Y^a \quad (X+Y)^a\,]} B$$

is an exact sequence of homogeneous B-module homomorphisms and if

$$\mathrm{HB}_a = \begin{bmatrix} H_{1,1} & H_{1,2} \\ H_{2,1} & H_{2,2} \\ H_{3,1} & H_{3,2} \end{bmatrix},$$

then $H_{i,j}$ is a homogeneous form in B of degree d_j; furthermore the signed maximal order minors of HB_a are

(5.2)
$$X^a = \det \begin{bmatrix} H_{2,1} & H_{2,2} \\ H_{3,1} & H_{3,2} \end{bmatrix}, \; Y^a = -\det \begin{bmatrix} H_{1,1} & H_{1,2} \\ H_{3,1} & H_{3,2} \end{bmatrix}, \text{ and}$$
$$(X+Y)^a = \det \begin{bmatrix} H_{1,1} & H_{1,2} \\ H_{2,1} & H_{2,2} \end{bmatrix}.$$

Each of the homogeneous forms $H_{i,j}$ has degree d_j. We say that column j of HB_a is a relation on $[\,X^a \quad Y^a \quad (X+Y)^a\,]$ of degree d_j. A quick look at (5.2) shows that the sum of the degrees d_1+d_2 must equal a. If $|d_1-d_2|\le 1$, then the Hilbert-Burch matrix HB_a is called *balanced*; otherwise, HB_a is called *unbalanced*.

We notice that the Hilbert-Burch matrix HB_a continues to make sense even if $a=p^e$, where $p>0$ is the characteristic of $\boldsymbol{k}$ and $e\ge 1$ is an integer. In this case,

$$\mathrm{HB}_a=\begin{bmatrix} 1 & -Y^a\\ 1 & X^a\\ -1 & 0\end{bmatrix},$$

the formulas of (5.2) hold, and HB_a is unbalanced.

The Hilbert-Burch matrix HB_a gives information which is relevant to the $R_{\boldsymbol{k},n}$ modules $Q_{\boldsymbol{k},n,N}$ by way of the $\boldsymbol{k}$-algebra homomorphism $\alpha\colon \boldsymbol{k}[X,Y]\to R_{\boldsymbol{k},n}$. The map α is defined by $\alpha(X)=x^n$ and $\alpha(Y)=y^n$. We see that

$$\alpha(X+Y)=x^n+y^n=-z^n.$$

Observation 5.3. *Let $\boldsymbol{k}$ be a field and n and N be positive integers. If n divides N, then $Q_{\boldsymbol{k},n,N}$ has finite projective dimension as an $R_{\boldsymbol{k},n}$-module.*

Proof. Let $N=an$ and let HB_a be a Hilbert-Burch matrix for $[X^a,Y^a,(X+Y)^a]$ over $\boldsymbol{k}$. If $\alpha\colon \boldsymbol{k}[X,Y]\to R_{\boldsymbol{k},n}$ is the $\boldsymbol{k}$-algebra homomorphism defined by $\alpha(X)=x^n$ and $\alpha(Y)=y^n$, then the ideal generated by the maximal order minors of $\alpha(\mathrm{HB}_a)$ is (X^N,Y^N,Z^N) and

$$0\to R_{\boldsymbol{k},n}^2\xrightarrow{\alpha(\mathrm{HB}_a)} R_{\boldsymbol{k},n}^3\to R_{\boldsymbol{k},n}$$

is a finite free resolution of $Q_{\boldsymbol{k},n,N}$. $\square$

Lemma 5.4. *Fix a positive integer a and a field $\boldsymbol{k}$. Suppose that some minimal, non-zero, homogeneous relation on $[X^a,Y^a,(X+Y)^a]$ in $\boldsymbol{k}[X,Y]$ has the form*

$$\xi=\begin{bmatrix} X\xi_1'\\ Y\xi_2'\\ (X+Y)\xi_3'\end{bmatrix},$$

where the ξ_i' are homogeneous elements of $\boldsymbol{k}[X,Y]$. If N and n are positive integers with $N=an+r$ for some r with $0\le r\le n$, then $\mathrm{pd}_{R_{\boldsymbol{k},n}} Q_{\boldsymbol{k},n,N}$ is finite.

Proof. We complete ξ to form a homogeneous Hilbert-Burch matrix

$$\mathrm{HB}_a=\begin{bmatrix} X\xi_1' & \eta_1\\ Y\xi_2' & \eta_2\\ (X+Y)\xi_3' & \eta_3\end{bmatrix}$$

for $[X^a, Y^a, (X+Y)^a]$ in $\boldsymbol{k}[X,Y]$. The signed maximal order minors of HB_a in the sense of (5.2) are X^a, Y^a, and $(X+Y)^a$. Apply the $\boldsymbol{k}$-algebra homomorphism $\alpha\colon \boldsymbol{k}[X,Y] \to R_{\boldsymbol{k},n}$ with $\alpha(X) = x^n$ and $\alpha(Y) = y^n$. We see that the ideal generated by the maximal order minors of

$$\mathrm{hb} = \begin{bmatrix} x^{n-r}\alpha(\xi_1') & (yz)^r\alpha(\eta_1) \\ y^{n-r}\alpha(\xi_2') & (xz)^r\alpha(\eta_2) \\ -z^{n-r}\alpha(\xi_3') & (xy)^r\alpha(\eta_3) \end{bmatrix}$$

in $R_{\boldsymbol{k},n}$ is generated by (x^N, y^N, z^N); and therefore

$$0 \to R_{\boldsymbol{k},n}^2 \xrightarrow{\mathrm{hb}} R_{\boldsymbol{k},n}^3 \to R_{\boldsymbol{k},n}$$

is a finite free resolution of $Q_{\boldsymbol{k},n,N}$. The entries of hb are in $R_{\boldsymbol{k},n}$. The only constraint on r is that r and $n-r$ both must be non-negative integers. $\square$

Lemma 5.5. *Let $\boldsymbol{k}$ be a field, n and a be positive integers, and R be the diagonal hypersurface ring $R_{\boldsymbol{k},n}$ of* (0.1)*.*

(1) *If some Hilbert-Burch matrix* HB_a *has the form*

$$\mathrm{HB}_a = \begin{bmatrix} F & XI \\ YG & J \\ (X+Y)H & K \end{bmatrix},$$

for homogeneous forms F,G,H,I,J,K in $B = \boldsymbol{k}[X,Y]$, then the R-module $Q_{\boldsymbol{k},n,N}$ has finite projective dimension for all $N = na + r$, with $0 \le r \le n$.

(2) *If some Hilbert-Burch matrix* HB_a *has the form*

$$\mathrm{HB}_a = \begin{bmatrix} Y(X+Y)F & I \\ (X+Y)G & XJ \\ YH & XK \end{bmatrix},$$

for homogeneous forms F,G,H,I,J,K in $B = \boldsymbol{k}[X,Y]$, then the R-module $Q_{\boldsymbol{k},n,N}$ has finite projective dimension for all $N = na - r$, with $0 \le r \le n$.

Proof. For the first assertion, the maximal order minors of the matrix

$$\mathrm{hb} = \begin{bmatrix} \alpha(F) & y^r z^r x^{n-r}\alpha(I) \\ x^r y^{n-r}\alpha(G) & z^r\alpha(J) \\ -x^r z^{n-r}\alpha(H) & y^r\alpha(K) \end{bmatrix}$$

generate the ideal $(x^{na+r}, y^{na+r}, z^{na+r})$ of R. For the second assertion, the maximal order minors of the matrix

$$\mathrm{hb} = \begin{bmatrix} x^r y^{n-r} z^{n-r}\alpha(F) & \alpha(I) \\ z^{n-r}\alpha(G) & x^{n-r}y^r\alpha(J) \\ y^{n-r}\alpha(H) & -x^{n-r}z^r\alpha(K) \end{bmatrix}$$

generate the ideal $(x^{na-r}, y^{na-r}, z^{na-r})$ of R. In each case

$$0 \to R^3 \xrightarrow{\mathrm{hb}} R^2 \to R$$

is a resolution of $Q_{\boldsymbol{k},n,N}$ by free R-modules. □

Theorem 5.6. *Consider the data $(\boldsymbol{k}, n, N)$ with $N = an+r$ and $0 \le r \le n$. If HB_a is unbalanced over $\boldsymbol{k}$, then the $R_{\boldsymbol{k},n}$-module $Q_{\boldsymbol{k},n,N}$ has finite projective dimension.*

Proof. Let

$$\mathrm{HB}_a = \begin{bmatrix} F & I \\ G & J \\ H & K \end{bmatrix}, \qquad \mathrm{HB}_{a+1} = \begin{bmatrix} M & Q \\ N & R \\ P & S \end{bmatrix},$$

$d_a = \deg(F)$, $D_a = \deg(I)$, $d_{a+1} = \deg(M)$, and $D_{a+1} = \deg(Q)$. The hypothesis that HB_a is unbalanced guarantees that $|D_a - d_a| \ge 2$. Note that

$$[XM, YN, (X+Y)P]^{\mathrm{T}}$$

is a relation on $[X^a, Y^a, (X+Y)^a]$, and thus we have

$$\begin{bmatrix} XM \\ YN \\ (X+Y)P \end{bmatrix} = f_1 \begin{bmatrix} F \\ G \\ H \end{bmatrix} + g_1 \begin{bmatrix} I \\ J \\ K \end{bmatrix} \tag{5.7}$$

for some $f_1, g_1 \in \boldsymbol{k}[X,Y]$, which can be taken to be homogeneous. Similarly we have

$$\begin{bmatrix} XQ \\ YR \\ (X+Y)S \end{bmatrix} = f_2 \begin{bmatrix} F \\ G \\ H \end{bmatrix} + g_2 \begin{bmatrix} I \\ J \\ K \end{bmatrix}, \tag{5.8}$$

for some homogeneous $f_2, g_2 \in \boldsymbol{k}[X,Y]$. Using equations (5.7) and (5.8) to calculate the 2×2 minors of the matrix HB_{a+1} yields

$$f_1 g_2 - g_1 f_2 = XY(X+Y). \tag{5.9}$$

Comparing degrees in (5.7) and (5.8) yields

$$\deg(f_1) - \deg(g_1) = \deg(f_2) - \deg(g_2) = D_a - d_a. \tag{5.10}$$

We first consider the case where all four polynomials f_1, g_1, f_2, g_2 are nonozero. Then (5.9) implies $\deg(f_1) + \deg(g_2) = \deg(g_1) + \deg(f_2) = 3$. There are two possibilities: either one of f_1, f_2, g_1, g_2 is a unit, in which case one can modify HB_a so that it has a column of the form $[XM, YN, (X+Y)P]^{\mathrm{T}}$ (which gives the conclusion by Lemma 5.4), or $\deg(f_1), \deg(g_1), \deg(f_2), \deg(g_2) \in \{1, 2\}$, which implies that $|\deg(F) - \deg(I)| = 0$ or 1, contradicting the hypothesis.

It remains to consider the case when one of f_1, g_1, f_2, g_2 is zero. Without loss of generality, assume $g_1 = 0$, and none of the polynomials f_1, f_2, or g_2 is a unit. Then (5.9) becomes $f_1 g_2 = XY(X+Y)$. Since X, Y, and $X+Y$ are irreducible in the Unique Factorization Domain $\boldsymbol{k}[X,Y]$, we may assume without loss of generality that one of the following holds:

(1) case 1: $f_1 = X, g_2 = Y(X+Y)$
(2) case 2: $f_1 = Y(X+Y), g_2 = X$

The hypothesis $|\deg(F) - \deg(I)| \geq 2$, together with the equation (5.10), ensures that $|\deg(f_2) - \deg(g_2)| \geq 2$. We have $\deg g_2 \in \{1, 2\}$ and f_2 is not a unit; so either f_2 is zero or $\deg f_2 \geq 3$. Write $f_2 = Xu + cY^s$ for some homogeneous $u \in \boldsymbol{k}[X,Y]$ and $c \in \boldsymbol{k}$, with $\deg f_2 = \deg u + 1 = s$. We have seen that if $c \neq 0$, then $s \geq 3$.

Consider case 1. Equations (5.7) and (5.8) give $Y|G$ and $(X+Y)|H$ in $\boldsymbol{k}[X,Y]$. We will apply part (1) of Lemma 5.5 to some appropriate modification of HB_a. Equation (5.8) tells us that

$$XQ = (Xu + cY^s)F + Y(X+Y)I.$$

Thus,

$$X(Q - uF - YI) = Y^2(cY^{s-2}F + I)$$

and X divides $cY^{s-2}F + I$ in $\boldsymbol{k}[X,Y]$. (Recall that the expression cY^{s-2} is meaningful in $\boldsymbol{k}[X,Y]$ because either $c = 0$ or $s \geq 3$.) We replace the column $[I, J, K]^{\mathrm{T}}$ in HB_a by $[I, J, K]^{\mathrm{T}} + cY^{s-2}[F, G, H]^{\mathrm{T}}$. Note that this new matrix can be used in place of HB_a, and it has the form specified in the first part of Lemma 5.5. Thus, the conclusion follows from Lemma 5.5.

Now consider case 2. Equation (5.7) implies that M is divisible by $Y(X+Y)$, N is divisible by $X+Y$, P is divisible by Y. We will apply part (2) of Lemma 5.5 to some appropriate modification of HB_{a+1}. Recall that (5.8) gives

$$\begin{bmatrix} XQ \\ YR \\ (X+Y)S \end{bmatrix} = (cY^s + uX)\begin{bmatrix} F \\ G \\ H \end{bmatrix} + X\begin{bmatrix} I \\ J \\ K \end{bmatrix}$$

and (5.7) gives

$$\begin{bmatrix} XM \\ YN \\ (X+Y)P \end{bmatrix} = (Y^2+XY)\begin{bmatrix} F \\ G \\ H \end{bmatrix}.$$

We see that (5.8) minus cY^{s-2} times (5.7) is

$$\begin{bmatrix} X & 0 & 0 \\ 0 & Y & 0 \\ 0 & 0 & X+Y \end{bmatrix}\left(\begin{bmatrix} Q \\ R \\ S \end{bmatrix} - cY^{s-2}\begin{bmatrix} M \\ N \\ P \end{bmatrix}\right) = X\left(u\begin{bmatrix} F \\ G \\ H \end{bmatrix} + \begin{bmatrix} I \\ J \\ K \end{bmatrix} - cY^{s-1}\begin{bmatrix} F \\ G \\ H \end{bmatrix}\right).$$

Thus, the bottom two entries of

$$\begin{bmatrix} Q \\ R \\ S \end{bmatrix} - cY^{s-2}\begin{bmatrix} M \\ N \\ P \end{bmatrix}$$

are divisible by X. We may replace the column $[Q,R,S]^{\mathrm{T}}$ in HB_{a+1} by $[Q,R,S]^{\mathrm{T}} - cY^{s-2}[M,N,P]^{\mathrm{T}}$. This new matrix can be used in the role of HB_{a+1}, and it has the form specified in the second part of Lemma 5.5. Thus, the conclusion follows from Lemma 5.5. □

Theorem 5.11 (Brenner and Kaid). *Let $\boldsymbol{k}$ be a field of positive characteristic p and let a be a positive integer.*

1. *If a is odd, then HB_a is unbalanced over $\boldsymbol{k}$ if and only if there exists an odd integer J and a power $q = p^e$ of p such that $|a - Jq| < \frac{q-1}{3}$.*
2. *If a is even, then HB_a is unbalanced over $\boldsymbol{k}$ if and only if there exists an odd integer J and a power $q = p^e$ of p such that $|a - Jq| < \frac{q}{3}$.*

Proof. Let $\bar{\boldsymbol{k}}$ be the algebraic closure of $\boldsymbol{k}$ and let A be the $\bar{\boldsymbol{k}}$-algebra

$$A = \bar{\boldsymbol{k}}[X,Y,Z]/(X^a, Y^a, Z^a).$$

It is shown in [1, Cor. 2.2] that A has the weak Lefschetz property (WLP) if and only if HB_a over $\bar{\boldsymbol{k}}$ is balanced. In [1, Thm. 2.6] numerical conditions, which characterize the set of a for which A does not have the WLP, are given. We have reformulated these numerical conditions. Notice that

$$\frac{3a+\epsilon}{6k+4} < q < \frac{3a-\epsilon}{6k+2} \iff (2k+1)q - \frac{q-\epsilon}{3} < a < (2k+1)q + \frac{q-\epsilon}{3}.$$

We notice that $\bar{\boldsymbol{k}}[X,Y]$ is a free $\boldsymbol{k}[X,Y]$-module. If $\mathbb{F}$ is a resolution of

$$\boldsymbol{k}[X,Y]/(X^a, Y^a, (X+Y)^a)$$

by free $\boldsymbol{k}[X,Y]$-modules, then $\mathbb{F}\otimes_{\boldsymbol{k}[X,Y]}\bar{\boldsymbol{k}}[X,Y]$ is a resolution of

$$\bar{\boldsymbol{k}}[X,Y]/(X^a,Y^a,(X+Y)^a)$$

by free $\bar{\boldsymbol{k}}[X,Y]$-modules. In particular, if HB_a is a Hilbert-Burch matrix for the matrix $[X^a,Y^a,(X+Y)^a]$ over $\boldsymbol{k}$, then HB_a is also a Hilbert-Burch matrix for $[X^a,Y^a,(X+Y)^a]$ over $\bar{\boldsymbol{k}}$. In other words, HB_a is balanced over $\boldsymbol{k}$ if and only if HB_a is balanced over $\bar{\boldsymbol{k}}$. $\square$

Lemma 5.12. *Fix a non-negative integer a and a field $\boldsymbol{k}$. Let n and N be positive integers with $N=an+r$ and $0\le r\le n$. Suppose that*

$$\xi=\begin{bmatrix} X^j\xi_1' \\ Y^j\xi_2' \\ (X+Y)^j\xi_3'\end{bmatrix}$$

is a non-zero, homogeneous relation on $[X^a,Y^a,(X+Y)^a]$ in $\boldsymbol{k}[X,Y]$ for some homogeneous polynomials $\xi_i'\in\boldsymbol{k}[X,Y]$.

(1) *If $j=1$ and $\deg\xi\le\lfloor\frac{a}{2}\rfloor$, then $\mathrm{pd}_{R_{\boldsymbol{k},n}}Q_{\boldsymbol{k},n,N}$ is finite.*
(2) *If $j=2$ and $\deg\xi\le\lceil\frac{a}{2}\rceil$, then $\mathrm{pd}_{R_{\boldsymbol{k},n}}Q_{\boldsymbol{k},n,N}$ is finite.*

Proof. (1) If ξ is a minimal relation, then use Lemma 5.4. If ξ is not a minimal relation, then HB_a is unbalanced; so use Theorem 5.6.

(2) If ξ is a minimal relation on $\rho_a=[X^a,Y^a,(X+Y)^a]$, then the conclusion follows from Lemma 5.4. Henceforth, we assume that ξ is not a minimal relation. The Hilbert-Burch Theorem guarantees that some minimal relation on ρ_a has degree at least $\lceil\frac{a}{2}\rceil$. Thus, ξ is a multiple of some minimal relation on ρ_a. In other words, $\xi=A\chi$ for some homogeneous A in $\boldsymbol{k}[X,Y]$ and some minimal relation χ. If degree $A=1$, then the conclusion follows from Lemma 5.4 applied to χ. If $\deg A\ge 2$, then HB_a is unbalanced and the conclusion follows from Theorem 5.6. $\square$

Observation 5.13. *Let J be an odd integer, $\boldsymbol{k}$ be a field of characteristic $p\ge 5$, $q=p^e$ for some integer $e\ge 1$, and $N=an+r$ for $0\le r\le n$.*

(1) *If $q\equiv 2 \mod 3$ and $a=Jq-\frac{q+1}{3}$, then $\mathrm{pd}_{R_{\boldsymbol{k},n}}Q_{\boldsymbol{k},n,N}$ is finite.*
(2) *If $q\equiv 1 \mod 3$ and $a=Jq-\frac{q-1}{3}$, then $\mathrm{pd}_{R_{\boldsymbol{k},n}}Q_{\boldsymbol{k},n,N}$ is finite.*

Proof. (1) Write J as $2k+1$. Select a relation $[\,A,B,C\,]^{\mathrm{T}}$ on $[X^J,Y^J,(X+Y)^J]$ of degree $\le k$. Observe that

$$\xi=\begin{bmatrix} X^{\frac{q+1}{3}}A^q \\ Y^{\frac{q+1}{3}}B^q \\ (X+Y)^{\frac{q+1}{3}}C^q\end{bmatrix}$$

is a relation on $[X^a, Y^a, (X+Y)^a]$ of degree at most $\frac{q+1}{3} + qk$. We see that

$$a = qJ - \frac{q+1}{3} = q(2k+1) - \frac{q+1}{3} = 2qk + \frac{2q-1}{3} = 2\left(kq + \frac{q-2}{3}\right) + 1.$$

So, $\deg \xi \le \lceil \frac{a}{2} \rceil$. Apply Lemma 5.12. Notice that $\frac{q+1}{3} \ge 2$ since $q \ge 5$.

(2) Observe that

$$\xi = \begin{bmatrix} X^{\frac{q-1}{3}} A^q \\ Y^{\frac{q-1}{3}} B^q \\ (X+Y)^{\frac{q-1}{3}} C^q \end{bmatrix}$$

is a relation on $[X^a, Y^a, (X+Y)^a]$ of degree at most $\frac{q-1}{3} + qk = \lfloor \frac{a}{2} \rfloor$. $\square$

Theorem 5.14. *Let $\boldsymbol{k}$ be a field of characteristic c and let n and N be positive integers. Write $N = an+r$ with $0 \le r \le n$. If $a \in T_c$, then the $R_{\boldsymbol{k},n}$-module $Q_{\boldsymbol{k},n,N}$ has finite projective dimension.*

Proof. The set T_0 is empty; so $c = p$ is positive. Lemma 5.15 establishes the result for $p = 2$. Throughout the rest of the proof we take $p \ge 3$.

Identify an odd integer J and a power $q = p^e$ of p so that a is close to Jq in the sense of Definition 1.10. First take $p = 3$. If $Jq - \frac{q}{3} + 1 \le a \le Jq + \frac{q}{3} - 1$, then $|Jq - a| \le \frac{q}{3} - 1 < \frac{q-1}{3}$; so, Theorem 5.11 yields that HB_a is unbalanced and the conclusion follows from Theorem 5.6. We now assume that $a = Jq - \frac{q}{3}$. Write $J = 2k+1$. Let $[A, B, C]^{\mathrm{T}}$ be a relation on $[X^J, Y^J, (X+Y)^J]$ of degree at most k. Observe that

$$\xi = \begin{bmatrix} A^q X^{q/3} \\ B^q Y^{q/3} \\ C^q (X+Y)^{q/3} \end{bmatrix}$$

is a relation on $[X^a, Y^a, (X+Y)^a]$ of degree at most $kq + \frac{q}{3} = \frac{a}{2}$. If ξ is a minimal relation on $[X^a, Y^a, (X+Y)^a]$, then $1 \le \frac{q}{3}$ and Lemma 5.4 yields the conclusion. Otherwise, HB_a is unbalanced at we may apply Theorem 5.6.

Henceforth, we consider $p \ge 5$. First we assume that $q \equiv 2 \mod 3$. If

$$Jq - \frac{q+1}{3} + 1 \le a \le Jq + \frac{q-2}{3},$$

then $|Jq - a| \le \frac{q-2}{3}$ and Theorem 5.11 shows that HB_a is unbalanced over $\boldsymbol{k}$ and Theorem 5.6 shows that $Q_{\boldsymbol{k},n,N}$ has finite projective dimension. If $a = Jq - \frac{q+1}{3}$, then apply Observation 5.13. Now we assume that $q \equiv 1 \mod 3$. If

$$Jq - \frac{q-1}{3} + 1 \le a \le Jq + \frac{q-4}{3},$$

then $|Jq - a| \le \frac{q-4}{3}$ and the conclusion follows from Theorems 5.11 and 5.6. If $a = Jq - \frac{q-1}{3}$, then we apply Observation 5.13. $\square$

Lemma 5.15. *Let $\boldsymbol{k}$ be a field of characteristic 2 and n and N be positive integers. If $n \le N$, then $Q_{\boldsymbol{k},n,N}$ has finite projective dimension as a module over $R_{\boldsymbol{k},n}$*

Proof. Write N in the form $N = qn + r$ for integers q and r with $q = 2^e$ and $0 \le r \le qn$. Consider the matrix

$$\text{hb} = \begin{bmatrix} y^r z^r & x^{N-2r} \\ x^r z^r & y^{N-2r} \\ x^r y^r & z^{N-2r} \end{bmatrix}.$$

Observe that the ideal generated by the maximal order minors of hb is (x^N, y^N, z^N). It follows that

$$0 \to R^2_{\boldsymbol{k},n} \xrightarrow{\text{hb}} R^3_{\boldsymbol{k},n} \to R_{\boldsymbol{k},n}$$

is a finite resolution of $Q_{\boldsymbol{k},n}$ by free $R_{\boldsymbol{k},n}$-modules. $\square$

Section 6. The set of non-negative integers may be partitioned as $S_p \cup T_p$.

Theorem 6.1. *If c is the characteristic of a field, then the set of non-negative integers is the disjoint union of S_c and T_c.*

Proof. The assertion is obvious if c is equal to 0 or 2. Henceforth, we take c to be a prime integer $p \ge 3$.

Let a be a non-negative integer and $\boldsymbol{k}$ be a field of characteristic p. Pick n and N so that $N = an + r$ with $1 \le r \le n-1$. The projective dimension of $Q_{\boldsymbol{k},n,N}$ over $R_{\boldsymbol{k},n}$ is either finite or infinite. If $a \in S_p$, then Theorem 3.5 shows that $\operatorname{pd} Q_{\boldsymbol{k},n,N}$ is infinite. If $a \in T_p$, then Theorem 5.14 shows that $\operatorname{pd} Q_{\boldsymbol{k},n,N}$ is finite. Thus the sets S_p and T_p are disjoint.

Suppose now that a is odd. We saw in Remark 1.2, Remark 1.13, and Observation 1.14 that if $p = 3$, then

$$a \in S_p \iff a-1 \in S_p \quad \text{and} \quad a \in T_p \iff a-1 \in T_p;$$

and if $p \ge 5$, then

$$a \in T_p \iff a+1 \in T_p \quad \text{and} \quad a \in T_p \iff a+1 \in T_p.$$

Thus, it suffices that prove that every non-negative even integer is in $S_p \cup T_p$ for each prime $p \ge 3$.

Let $p \ge 5$ be a prime integer and let m be an even integer with $m \notin S_p$. We prove that $m \in T_p$. Write m as $a_0 + a_1 p + \ldots + a_t p^t$ with $a_0, \ldots, a_t$ digits in the sense of Notation 1.5. Since m is even and we are assuming $m \notin S_p$, it follows that at least two of the digits, say a_k and a_ℓ are odd and less that $p/3$. More precisely, assume that k is the smallest index such that a_k is odd, and there are an odd number of indices among $a_{k+1}, \dots, a_t$ for which the corresponding digits are odd.

We can write

$$m = p^{k+1}J + a_k p^k + a_{k-1}p^{k-1} + \dots + a_0$$

where $J = a_{k+1} + \dots + a_t p^{t-k-1}$ is an odd integer. Let $q = p^{k+1}$. We obtain an upper bound for $|m - Jq|$. Let u denote the largest odd integer which is less than $p/3$, and let v denote the largest even integer which is less than $2p/3$. Since $a_k, a_{k-1}, \dots, a_0$ are digits, and a_k is odd, we have

$$\begin{aligned} |m - Jq| &= |a_k p^k + a_{k-1}p^{k-1} + \dots + a_0| \\ &< up^k + v(p^{k-1} + \dots + 1) \\ &= up^k + v\frac{p^k - 1}{p-1} < \left(u + \frac{v}{p-1}\right)p^k. \end{aligned}$$

If $p \equiv 1 \mod 3$, then $u = (p-4)/3$ and $v = (2p-2)/3$. If $p \equiv 2 \mod 3$, then $u = (p-2)/3$ and $v = (2p-4)/3$. A straightforward calculation establishes

$$u + \frac{v}{p-1} < \frac{p}{3}$$

in each case. We conclude that $|m - Jq| < (\frac{p}{3})p^k = \frac{q}{3}$; and therefore $m \in T_p$ by Observation 1.14.

Now take $p = 3$. We prove that every non-negative even integer is in $S_3 \cup T_3$. We know that $0 \in S_3$ and $2 \in T_3$. The proof proceeds by induction. Suppose that $m \geq 4$ is an even integer with $m \notin T_3$. Suppose further that every non-negative even integer a with $a < m$ is in $S_3 \cup T_3$. We observe first that m is not congruent to $2 \mod 3$. Indeed, if $m \equiv 2 \mod 3$. Then $m + 1$ is equal to $3J$ for some odd integer J; thus, $m + 1 \in T_3$ by the definition of T_3 and $m \in T_3$ by Remark 1.13 and this is a contradiction. Thus, $m \equiv 0$ or $m \equiv 1 \mod 3$. We treat the two cases separately.

We first suppose that $m \equiv 0 \mod 3$. In this case, we consider the even integer $a = \frac{m}{3}$. We know that $0 \leq a < m$; so the induction hypothesis shows that $a \in S_3 \cup T_3$. On the other hand, it is not possible for $a \in T_3$. Indeed if $a \in T_3$, then there exists an odd integer J and a power $q = 3^e$ with $Jq - \frac{q}{3} \leq a \leq Jq + \frac{q}{3} - 1$. Multiply by 3 to see that

$$J(3q) - \tfrac{3q}{3} \leq m \leq J(3q) + \tfrac{3q}{3} - 3 < J(3q) + \tfrac{3q}{3} - 1;$$

and therefore $m \in T_3$, which is a contradiction. We have $a \in S_3 \cup T_3$ and $a \notin T_3$. It follows that a is an even integer in S_3. The definition of S_3 shows that $m = 3a \in S_3$.

Finally, we suppose that $m \equiv 1 \mod 3$. In this case, we consider the even integer $a = \frac{m-4}{3}$. We know that $0 \le a < m$; so the induction hypothesis shows that a is in $S_3 \cup T_3$. On the other hand, it is not possible for $a \in T_3$. Indeed if $a \in T_3$, then there exists an odd integer J and a power $q = 3^e$ with $Jq - \frac{q}{3} \le a \le Jq + \frac{q}{3} - 1$. Multiply by 3 and add 4 to see that

$$J(3q) - \tfrac{3q}{3} < J(3q) - \tfrac{3q}{3} + 4 \le m \le J(3q) + \tfrac{3q}{3} + 1.$$

The integer m is even and $\equiv 1 \mod 3$; whereas $J(3q) + \frac{3q}{3} + 1$ is odd and $\equiv 1 \mod 3$. It follows that $m \le J(3q) + \frac{3q}{3} - 2$. At any rate, we would have

$$J(3q) - \tfrac{3q}{3} \le m \le J(3q) + \tfrac{3q}{3} - 2 < J(3q) + \tfrac{3q}{3} - 1;$$

and therefore $m \in T_3$, which is a contradiction. We have $a \in S_3 \cup T_3$ and $a \notin T_3$. It follows that a is an even integer in S_3. The definition of S_3 shows that m, which is equal to $3a + 4$, is in S_3. $\square$

Theorem 6.2. *Let $\boldsymbol{k}$ be a field of characteristic c and let n and N be positive integers. Write N in the form $N = \theta n + r$, for integers θ and r, with $0 \le r \le n-1$. Then*

$$\operatorname{pd}_{R_{\boldsymbol{k},n}} Q_{\boldsymbol{k},n,N} = \infty \iff \theta \in S_c \text{ and } 1 \le r.$$

Proof. Corollary 3.6 establishes "($\Leftarrow$)". Theorem 6.1 shows that if $\theta \notin S_c$, then θ is in T_c and Theorem 5.14 shows that if $\theta \in T_c$, then $\operatorname{pd}_{R_{\boldsymbol{k},n}} Q_{\boldsymbol{k},n,N}$ is finite. Furthermore, Observation 5.3 shows that if $r = 0$, then $\operatorname{pd}_{R_{\boldsymbol{k},n}} Q_{\boldsymbol{k},n,N}$ is finite. $\square$

We rephrase Theorem 6.2 without appealing directly to the sets S_c and T_c. Recall the meaning of the operation $\{\ \}$ from (0.3).

Theorem 6.3. *Let $\boldsymbol{k}$ be a field of characteristic c and let n and N be positive integers. Then $\operatorname{pd}_{R_{\boldsymbol{k},n}} Q_{\boldsymbol{k},n,N}$ is finite if and only if at least one of the following conditions hold:*

1. *n divides N, or*
2. *$c = 2$ and $n \le N$, or*
3. *$c = p$ is an odd prime and there exist an odd integer J and a power $q = p^e$ of p with $e \ge 1$ and $|Jq - \frac{N}{n}| < \{\frac{q}{3}\}$.*

In particular, if $c = 0$, then $\operatorname{pd}_{R_{\boldsymbol{k},n}} Q_{\boldsymbol{k},n,N}$ is finite if and only if n divides N.

Proof. According to Theorem 6.2, $\operatorname{pd} Q_{\boldsymbol{k},n,N}$ is finite if and only if $\lfloor \frac{N}{n} \rfloor \in T_c$ or $n|N$. Definition 1.10 shows that T_0 is empty, T_2 is the set of all positive integers, and if $p \ge 3$, then

$$\lfloor \tfrac{N}{n} \rfloor \in T_p \iff Jq - \{\tfrac{q}{3}\} \le \lfloor \tfrac{N}{n} \rfloor < Jq + \{\tfrac{q}{3}\},$$

for some J and q as described in the statement of the result. The case $n|N$ has been taken care of in (1). When we restrict our attention to $n \not| N$, then $\lfloor \frac{N}{n} \rfloor < \frac{N}{n}$ and

$$\lfloor \tfrac{N}{n} \rfloor \in T_p \iff Jq - \{\tfrac{q}{3}\} \le \lfloor \tfrac{N}{n} \rfloor < \tfrac{N}{n} < Jq + \{\tfrac{q}{3}\} \iff |Jq - \tfrac{N}{n}| < \{\tfrac{q}{3}\}. \quad \square$$

Section 7. The proof of Lemma 3.4.

Recall, from Definition 3.2, that if d, a, and b are non-negative integers then

$$\operatorname{Poly}_{d,a,b}(A,B) = \sum_{i=0}^{d} (-1)^i \binom{a+d-i}{a} \binom{b+i}{b} A^{d-i} B^i$$

in the polynomial ring $\mathbb{Z}[A,B]$. In this section we prove Lemma 3.4.

Lemma 3.4. *For each positive integer δ, the polynomials*

$$P_{2\delta-1}(A,B,C) = \begin{cases} (-1)^{\delta} A \operatorname{Poly}_{\delta-1,\delta,\delta-1}(A,B) \operatorname{Poly}_{\delta-1,\delta,\delta-1}(A,C) \\ +B \operatorname{Poly}_{\delta-1,\delta,\delta-1}(B,A) \operatorname{Poly}_{\delta-1,\delta,\delta-1}(A,C) \\ +C \operatorname{Poly}_{\delta-1,\delta,\delta-1}(A,B) \operatorname{Poly}_{\delta-1,\delta,\delta-1}(C,A) \\ +(-1)^{\delta+1} \binom{2\delta}{\delta} \binom{3\delta-1}{\delta-1} A^{2\delta-1} \end{cases}$$

and

$$P_{2\delta}(A,B,C) = \begin{cases} (-1)^{\delta+1} \operatorname{Poly}_{\delta,\delta,\delta}(A,B) \operatorname{Poly}_{\delta,\delta,\delta}(A,C) \\ +B \operatorname{Poly}_{\delta-1,\delta,\delta}(B,A) \operatorname{Poly}_{\delta,\delta,\delta}(A,C) \\ +C \operatorname{Poly}_{\delta,\delta,\delta}(A,B) \operatorname{Poly}_{\delta-1,\delta,\delta}(C,A) \\ +(-1)^{\delta} \binom{2\delta}{\delta} \binom{3\delta}{\delta} A^{2\delta} \end{cases}$$

are in the ideal $(A+B+C)\mathbb{Z}[A,B,C]$.

The proof of Lemma 3.4 is carried out in Lemmas 7.3 and 7.13. In Lemma 7.3 we exhibit the polynomial $Q_{2\delta-1}(A,B,C) \in \mathbb{Z}[A,B,C]$ such that

$$P_{2\delta-1}(A,B,C) = (A+B+C) Q_{2\delta-1}(A,B,C) \tag{7.1}$$

in $\mathbb{Z}[A,B,C]$. In Lemma 7.13 we express $P_{2\delta}(A,B,C)$ as an element of the ideal of $\mathbb{Z}[A,B,C]$ generated by $P_{2\delta-1}(A,B,C)$ and $A+B+C$; hence, $P_{2\delta}$ is also an element of the ideal $(A+B+C)\mathbb{Z}[A,B,C]$.

Definition 7.2.

(1) For integers d and δ, with $0 \le d$, define the polynomial

$$H_{d,\delta}(A,B) = \sum_{i=0}^{d} (-1)^i \binom{2\delta-1-i}{d-i}\binom{d+i}{d} A^{d-i}B^i$$

of $\mathbb{Z}[A,B,C]$.

(2) For each positive integer δ, define the polynomial $Q_{2\delta-1}(A,B,C)$ to be

$$(-1)^{\delta+1}\sum_{d=0}^{\delta-1} \frac{\binom{3\delta-1}{2\delta+d}\binom{2\delta-d-1}{\delta}(2d+1)}{\delta\binom{\delta-1}{d}^2} H_{d,\delta}(A,B)H_{d,\delta}(A,C)A^{2(\delta-d-1)}$$

in $\mathbb{Q}[A,B,C]$.

Lemma 7.3. *If δ is a positive integer, then the polynomial $Q_{2\delta-1}(A,B,C)$ is in $\mathbb{Z}[A,B,C]$ and equation (7.1) holds in $\mathbb{Z}[A,B,C]$.*

Proof. The polynomial $P_{2\delta-1}(A,B,C)$ is in $\mathbb{Z}[A,B,C]$, the polynomial $A+B+C$ generates a prime ideal of $\mathbb{Z}[A,B,C]$, $Q_{2\delta-1}(A,B,C)$ is in $\mathbb{Q}[A,B,C]$, and the ring $\mathbb{Z}[A,B,C]$ is a Unique Factorization Domain. Once we prove that equation (7.1) holds in $\mathbb{Q}[A,B,C]$, then it follows that $Q_{2\delta-1}(A,B,C)$ is actually in $\mathbb{Z}[A,B,C]$ and the equality (7.1) takes place in $\mathbb{Z}[A,B,C]$.

If $\delta = 1$, then $P_{2\delta-1}(A,B,C) = A+B+C$ and $Q_{2\delta-1}(A,B,C) = 1$; thus, equation (7.1) holds in this case. Henceforth, we assume that $2 \le \delta$.

To show that (7.1) holds in $\mathbb{Q}[A,B,C]$, we must show that

$$(*) = \begin{cases} A\,\mathrm{Poly}_{\delta-1,\delta,\delta-1}(A,B)\,\mathrm{Poly}_{\delta-1,\delta,\delta-1}(A,C) \\ +(-1)^\delta B\,\mathrm{Poly}_{\delta-1,\delta,\delta-1}(B,A)\,\mathrm{Poly}_{\delta-1,\delta,\delta-1}(A,C) \\ +(-1)^\delta C\,\mathrm{Poly}_{\delta-1,\delta,\delta-1}(A,B)\,\mathrm{Poly}_{\delta-1,\delta,\delta-1}(C,A) \\ +(A+B+C)\sum_{d=0}^{\delta-1} \frac{\binom{2\delta-d-1}{\delta}\binom{3\delta-1}{2\delta+d}(2d+1)}{\delta\binom{\delta-1}{d}^2} H_{d,\delta}(A,B)H_{d,\delta}(A,C)A^{2(\delta-d-1)} \end{cases}$$

equals $\binom{2\delta}{\delta}\binom{3\delta-1}{\delta-1}A^{2\delta-1}$. Observe that if $A^aB^bC^c$ appears in $(*)$, then $0 \le b, c \le \delta$, $0 \le a \le 2\delta-1$, and $a+b+c = 2\delta-1$. If S is a statement, then define

$$\chi(S) = \begin{cases} 1 & \text{if } S \text{ is true} \\ 0 & \text{if } S \text{ is false.} \end{cases}$$

Fix integers b, c, and δ, with $0 \le b, c \le \delta$ and $b+c \le 2\delta-1$. We see that the coefficient of $A^{2\delta-1-b-c}B^bC^c$ in $(*)$ is $(-1)^{b+c}$ times $(**)_{b,c,\delta}$, where $(**)_{b,c,\delta}$ is the rational number

$$(**)_{b,c,\delta} = D_{b,c,\delta} + \sum_{d=0}^{\delta-1} \frac{\binom{3\delta-1}{2\delta+d}\binom{2\delta-d-1}{\delta}(2d+1)}{\delta\binom{\delta-1}{d}^2} E_{b,c,d,\delta}, \tag{7.4}$$

for

$$D_{b,c,\delta}=\begin{cases}\ \ \chi(b,c\le\delta-1)\binom{2\delta-1-b}{\delta-1-b}\binom{\delta-1+b}{\delta-1}\binom{2\delta-1-c}{\delta-1-c}\binom{\delta-1+c}{\delta-1}\\ +\chi(1\le b)\chi(c\le\delta-1)\binom{2\delta-1-b}{\delta-1}\binom{\delta-1+b}{b-1}\binom{2\delta-1-c}{\delta-1-c}\binom{\delta-1+c}{\delta-1}\\ +\chi(b\le\delta-1)\chi(1\le c)\binom{2\delta-1-b}{\delta-1-b}\binom{\delta-1+b}{\delta-1}\binom{2\delta-1-c}{\delta-1}\binom{\delta-1+c}{c-1}\end{cases}$$

and

$$E_{b,c,d,\delta}=\begin{cases}\ \ \chi(b,c\le d)\binom{2\delta-1-b}{d-b}\binom{d+b}{d}\binom{2\delta-1-c}{d-c}\binom{d+c}{d}\\ -\chi(1\le b\le d+1)\chi(c\le d)\binom{2\delta-b}{d-b+1}\binom{d+b-1}{d}\binom{2\delta-1-c}{d-c}\binom{d+c}{d}\\ -\chi(b\le d)\chi(1\le c\le d+1)\binom{2\delta-1-b}{d-b}\binom{d+b}{d}\binom{2\delta-c}{d-c+1}\binom{d+c-1}{d}.\end{cases}$$

We complete the proof by showing that

$$(**)_{b,c,\delta}=\begin{cases}\binom{2\delta}{\delta}\binom{3\delta-1}{\delta-1} & \text{if } b=c=0\\ 0 & \text{otherwise.}\end{cases}\tag{7.5}$$

The data has been chosen to ensure that δ is not zero; so, $\frac{1}{\delta}$ is a well defined rational number. Use

$$\binom{2\delta-1-\gamma}{\delta-1-\gamma}=\tfrac{\delta-\gamma}{\delta}\binom{2\delta-1-\gamma}{\delta-1}\quad\text{and}\quad\binom{\delta-1+\gamma}{\gamma-1}=\tfrac{\gamma}{\delta}\binom{\delta-1+\gamma}{\delta-1},$$

for $\gamma\in\{b,c\}$, to transform $D_{b,c,\delta}$ into

$$\begin{cases}\chi(b,c\le\delta-1)\frac{(\delta-b)}{\delta}\binom{2\delta-1-b}{\delta-1}\binom{\delta-1+b}{\delta-1}\frac{(\delta-c)}{\delta}\binom{2\delta-1-c}{\delta-1}\binom{\delta-1+c}{\delta-1}\\ +\chi(1\le b)\chi(c\le\delta-1)\binom{2\delta-1-b}{\delta-1}\frac{b}{\delta}\binom{\delta-1+b}{\delta-1}\frac{(\delta-c)}{\delta}\binom{2\delta-1-c}{\delta-1}\binom{\delta-1+c}{\delta-1}\\ +\chi(b\le\delta-1)\chi(1\le c)\frac{(\delta-b)}{\delta}\binom{2\delta-1-b}{\delta-1}\binom{\delta-1+b}{\delta-1}\binom{2\delta-1-c}{\delta-1}\frac{c}{\delta}\binom{\delta-1+c}{\delta-1}.\end{cases}$$

Observe that if $\gamma\in\{b,c\}$, then

$$(\delta-\gamma)\chi(\gamma\le\delta-1)=(\delta-\gamma)\quad\text{and}\quad\gamma\chi(1\le\gamma)=\gamma.$$

Indeed, the ambient hypothesis $0\le b,c\le\delta$ ensures that $\chi(\gamma\le\delta-1)$ takes the value 1 unless $\gamma=\delta$, in which case the numbers $(\delta-\gamma)$ and $\chi(\gamma\le\delta-1)$ are both zero; and $\chi(1\le\gamma)$ takes the value 1 unless $\gamma=0$, in which case the numbers γ and $\chi(1\le\gamma)$ are both zero. Thus;

$$D_{b,c,\delta}=\left(\tfrac{(\delta-b)}{\delta}\tfrac{(\delta-c)}{\delta}+\tfrac{b}{\delta}\tfrac{(\delta-c)}{\delta}+\tfrac{(\delta-b)}{\delta}\tfrac{c}{\delta}\right)\binom{2\delta-1-b}{\delta-1}\binom{\delta-1+b}{\delta-1}\binom{2\delta-1-c}{\delta-1}\binom{\delta-1+c}{\delta-1};$$

and therefore,

$$D_{b,c,\delta}=\frac{(\delta^2-bc)}{\delta^2}\binom{2\delta-1-b}{\delta-1}\binom{\delta-1+b}{\delta-1}\binom{2\delta-1-c}{\delta-1}\binom{\delta-1+c}{\delta-1}.\tag{7.6}$$

The parameter δ is at least 2; so we may use the identities

$$\tbinom{2\delta-1-\gamma}{\delta-1}=\tfrac{\delta+1-\gamma}{\delta-1}\tbinom{2\delta-1-\gamma}{\delta-2}\quad\text{and}\quad\tbinom{\delta-1+\gamma}{\delta-1}=\tfrac{\delta}{\delta+\gamma}\tbinom{\delta+\gamma}{\delta},$$

for $\gamma\in\{b,c\}$, to write

$$D_{b,c,\delta}=\tfrac{(\delta^2-bc)(\delta+1-b)(\delta+1-c)}{(\delta-1)^2(\delta+b)(\delta+c)}\tbinom{2\delta-1-b}{\delta-2}\tbinom{\delta+b}{\delta}\tbinom{2\delta-1-c}{\delta-2}\tbinom{\delta+c}{\delta}. \tag{7.7}$$

In a similar manner, we simplify $E_{b,c,d,\delta}$, provided $\max\{b,c,1\}\le d\le\delta-1$. Use

$$\tbinom{2\delta-1-\gamma}{d-\gamma}=\tfrac{d-\gamma+1}{2\delta-1-d}\tbinom{2\delta-1-\gamma}{2\delta-2-d},\quad\tbinom{2\delta-\gamma}{d-\gamma+1}=\tfrac{2\delta-\gamma}{2\delta-1-d}\tbinom{2\delta-1-\gamma}{2\delta-2-d},$$

and $\tbinom{d+\gamma-1}{d}=\tfrac{\gamma}{d+\gamma}\tbinom{d+\gamma}{d}$, for $\gamma\in\{b,c\}$, to obtain

$$E_{b,c,d,\delta}=\begin{cases}\tfrac{d-b+1}{2\delta-1-d}\tbinom{2\delta-1-b}{2\delta-2-d}\tbinom{d+b}{d}\tfrac{d-c+1}{2\delta-1-d}\tbinom{2\delta-1-c}{2\delta-2-d}\tbinom{d+c}{d}\\ -\chi(1\le b)\tfrac{(2\delta-b)}{2\delta-1-d}\tbinom{2\delta-b-1}{2\delta-2-d}\tfrac{b}{d+b}\tbinom{d+b}{d}\tfrac{d-c+1}{2\delta-1-d}\tbinom{2\delta-1-c}{2\delta-2-d}\tbinom{d+c}{d}\\ -\chi(1\le c)\tfrac{d-b+1}{2\delta-1-d}\tbinom{2\delta-1-b}{2\delta-2-d}\tbinom{d+b}{d}\tfrac{(2\delta-c)}{2\delta-1-d}\tbinom{2\delta-c-1}{2\delta-2-d}\tfrac{c}{d+c}\tbinom{d+c}{d}.\end{cases}$$

The hypothesis $\max\{b,c,1\}\le d\le\delta-1$ ensures that the denominators $2\delta-1-d$, $d+b$, and $d+c$ all are non-zero. We still have $\gamma\chi(1\le\gamma)=\gamma$ for $\gamma\in\{b,c\}$. It follows that

$$E_{b,c,d,\delta}=\tfrac{E'_{b,c,d,\delta}}{(2\delta-1-d)^2(d+b)(d+c)}\tbinom{2\delta-1-b}{2\delta-2-d}\tbinom{d+b}{d}\tbinom{2\delta-1-c}{2\delta-2-d}\tbinom{d+c}{d} \tag{7.8}$$

for

$$E'_{b,c,d,\delta}=(d-b+1)(d+b)(d+c)(d-c+1)-(2\delta-b)b(d+c)(d-c+1)-(d-b+1)(2\delta-c)c(d+b),$$

provided $\max\{b,c,1\}\le d\le\delta-1$.

If $0\le r\le\delta$, then let

$$W_{b,c,r,\delta}=D_{b,c,d}+\sum_{d=r}^{\delta-1}\frac{\binom{3\delta-1}{2\delta+d}\binom{2\delta-d-1}{\delta}(2d+1)}{\delta\binom{\delta-1}{d}^2}E_{b,c,d,\delta}. \tag{7.9}$$

Our goal is given in (7.5); we must evaluate the rational number $(**)_{b,c,\delta}$, which is given in (7.4). Write

$$(**)_{b,c,\delta}=F_{b,c,\delta}+G_{b,c,\delta} \tag{7.10}$$

for

$$F_{b,c,\delta}=W_{b,c,\max\{b,c,1\},\delta}$$

and

$$G_{b,c,\delta}=\sum_{d=0}^{\max\{b,c,1\}-1}\frac{\binom{3\delta-1}{2\delta+d}\binom{2\delta-d-1}{\delta}(2d+1)}{\delta\binom{\delta-1}{d}^2}E_{b,c,d,\delta}.$$

We next show, by induction, that if $\max\{b,c,1\}\le r\le\delta$, then

$$W_{b,c,r,\delta}=\frac{\binom{3\delta-1}{2\delta+r-1}\binom{2\delta-1-r}{\delta-1}}{(2\delta-1-r)^2\binom{\delta}{r}^2}\frac{(r+1-b)(r+1-c)(r^2-bc)}{(r+b)(r+c)}\binom{2\delta-1-b}{2\delta-2-r}\binom{r+b}{r}\binom{2\delta-1-c}{2\delta-2-r}\binom{r+c}{r}. \tag{7.11}$$

If $r=\delta$, then (7.9) shows that $W_{b,c,\delta,\delta}=D_{b,c,\delta}$; thus, $W_{b,c,\delta,\delta}$ is given in (7.7). There is no difficulty in seeing that (7.7) is the same as the right side of (7.11) when $r=\delta$. Assume that (7.11) holds at $r+1$ and $\max\{b,c,1\}\le r<\delta$. Then, according to (7.9),

$$W_{b,c,r,\delta}=W_{b,c,r+1,\delta}+\frac{\binom{3\delta-1}{2\delta+r}\binom{2\delta-r-1}{\delta}(2r+1)}{\delta\binom{\delta-1}{r}^2}E_{b,c,r,\delta}. \tag{7.12}$$

We see, from (7.11), that $W_{b,c,r+1,\delta}$

$$=\frac{\binom{3\delta-1}{2\delta+r}\binom{2\delta-2-r}{\delta-1}}{(2\delta-2-r)^2\binom{\delta}{r+1}^2}\frac{(r+2-b)(r+2-c)((r+1)^2-bc)}{(r+1+b)(r+1+c)}\binom{2\delta-1-b}{2\delta-3-r}\binom{r+1+b}{r+1}\binom{2\delta-1-c}{2\delta-3-r}\binom{r+1+c}{r+1}.$$

Use $\binom{2\delta-1-\gamma}{2\delta-3-r}=\frac{2\delta-2-r}{r+2-\gamma}\binom{2\delta-1-\gamma}{2\delta-2-r}$ and $\binom{r+1+\gamma}{r+1}=\frac{r+1+\gamma}{r+1}\binom{r+\gamma}{r}$, for $\gamma\in\{b,c\}$, to see that

$$W_{b,c,r+1,\delta}=\frac{\binom{3\delta-1}{2\delta+r}\binom{2\delta-2-r}{\delta-1}}{\binom{\delta}{r+1}^2}\frac{((r+1)^2-bc)}{(r+1)^2}\binom{2\delta-1-b}{2\delta-2-r}\binom{r+b}{r}\binom{2\delta-1-c}{2\delta-2-r}\binom{r+c}{r}.$$

The identities $(r+1)\binom{\delta}{r+1}=\delta\binom{\delta-1}{r}$ and $\frac{1}{\delta}\binom{2\delta-2-r}{\delta-1}=\frac{1}{2\delta-1-r}\binom{2\delta-1-r}{\delta}$ allow us to transform $W_{b,c,r+1,\delta}$ into

$$W_{b,c,r+1,\delta}=\frac{\binom{3\delta-1}{2\delta+r}\binom{2\delta-1-r}{\delta}}{\delta\binom{\delta-1}{r}^2}\frac{((r+1)^2-bc)}{2\delta-1-r}\binom{2\delta-1-b}{2\delta-2-r}\binom{r+b}{r}\binom{2\delta-1-c}{2\delta-2-r}\binom{r+c}{r}.$$

Apply (7.12) and (7.8) to see that $W_{b,c,r,\delta}$

$$=\begin{cases}\frac{\binom{3\delta-1}{2\delta+r}\binom{2\delta-1-r}{\delta}}{\delta\binom{\delta-1}{r}^2}\frac{((r+1)^2-bc)}{2\delta-1-r}\binom{2\delta-1-b}{2\delta-2-r}\binom{r+b}{r}\binom{2\delta-1-c}{2\delta-2-r}\binom{r+c}{r}\\ +\frac{\binom{3\delta-1}{2\delta+r}\binom{2\delta-r-1}{\delta}(2r+1)}{\delta\binom{\delta-1}{r}^2}\frac{E'_{b,c,r,\delta}}{(2\delta-1-r)^2(r+b)(r+c)}\binom{2\delta-1-b}{2\delta-2-r}\binom{r+b}{r}\binom{2\delta-1-c}{2\delta-2-r}\binom{r+c}{r},\end{cases}$$

for

$$E'_{b,c,r,\delta}=(r-b+1)(r+b)(r+c)(r-c+1)-(2\delta-b)b(r+c)(r-c+1)-(r-b+1)(2\delta-c)c(r+b).$$

Thus, $W_{b,c,r,\delta}$

$$= \frac{\binom{3\delta-1}{2\delta+r}\binom{2\delta-1-r}{\delta}\binom{2\delta-1-b}{2\delta-2-r}\binom{r+b}{r}\binom{2\delta-1-c}{2\delta-2-r}\binom{r+c}{r}}{(2\delta-1-r)^2\delta\binom{\delta-1}{r}^2(r+b)(r+c)}\begin{cases}((r+1)^2-bc)(2\delta-1-r)(r+b)(r+c)\\ +(2r+1)E'_{b,c,r,\delta}\end{cases}$$

One may check that

$$((r+1)^2-bc)(2\delta-1-r)(r+b)(r+c)+(2r+1)E'_{b,c,r,\delta}=(r-b+1)(r-c+1)(r^2-bc)(2\delta+r);$$

hence, $W_{b,c,r,\delta}$ equals

$$\frac{\binom{3\delta-1}{2\delta+r}\binom{2\delta-1-r}{\delta}\binom{2\delta-1-b}{2\delta-2-r}\binom{r+b}{r}\binom{2\delta-1-c}{2\delta-2-r}\binom{r+c}{r}}{(2\delta-1-r)^2\delta\binom{\delta-1}{r}^2(r+b)(r+c)}(r-b+1)(r-c+1)(r^2-bc)(2\delta+r).$$

Apply the identities

$$\delta\binom{\delta-1}{r}=(\delta-r)\binom{\delta}{r},\quad \delta\binom{2\delta-1-r}{\delta}=(\delta-r)\binom{2\delta-1-r}{\delta-1},\quad \text{and}\quad (2\delta+r)\binom{3\delta-1}{2\delta+r}=(\delta-r)\binom{3\delta-1}{2\delta+r-1},$$

in order to see that

$$\frac{\binom{3\delta-1}{2\delta+r}\binom{2\delta-1-r}{\delta}(2\delta+r)}{\delta\binom{\delta-1}{r}^2}=\frac{\binom{3\delta-1}{2\delta+r-1}\binom{2\delta-1-r}{\delta-1}}{\binom{\delta}{r}^2}$$

and

$$W_{b,c,r,\delta}=\frac{\binom{3\delta-1}{2\delta+r-1}\binom{2\delta-1-r}{\delta-1}}{(2\delta-1-r)^2\binom{\delta}{r}^2}\frac{(r+1-b)(r+1-c)(r^2-bc)}{(r+b)(r+c)}\binom{2\delta-1-b}{2\delta-2-r}\binom{r+b}{r}\binom{2\delta-1-c}{2\delta-2-r}\binom{r+c}{r};$$

thereby completing the proof of (7.11).

We saw at (7.10) that $(**)_{b,c,\delta}=F_{b,c,\delta}+G_{b,c,\delta}$. All of the formulas are symmetric in b and c; so no harm is done if we only check (7.5) for $b\le c$. Apply (7.11) at

$$r=\max\{b,c,1\}=\begin{cases}1 & \text{if } b=c=0\\ c & b,1\le c\end{cases}$$

to see that

$$F_{b,c,\delta}=\begin{cases}\frac{4\binom{3\delta-1}{2\delta}\binom{2\delta-2}{\delta-1}}{(2\delta-2)^2\delta^2}\binom{2\delta-1}{2\delta-3}^2, & \text{if } b=c=0,\\ \frac{\binom{3\delta-1}{2\delta+c-1}\binom{2\delta-1-c}{\delta-1}}{(2\delta-1-c)^2\binom{\delta}{c}^2}\frac{(c+1-b)(c-b)}{2(c+b)}\binom{2\delta-1-b}{2\delta-2-c}\binom{c+b}{c}\binom{2\delta-1-c}{2\delta-2-c}\binom{2c}{c}, & \text{if } b,1\le c.\end{cases}$$

On the other hand,

$$G_{b,c,\delta}=\sum_{d=0}^{\max\{b,c,1\}-1}\frac{\binom{3\delta-1}{2\delta+d}\binom{2\delta-d-1}{\delta}(2d+1)}{\delta\binom{\delta-1}{d}^2}E_{b,c,d,\delta},$$

with

$$E_{b,c,d,\delta} = \begin{cases} \chi(b,c \le d)\binom{2\delta-1-b}{d-b}\binom{d+b}{d}\binom{2\delta-1-c}{d-c}\binom{d+c}{d} \\ -\chi(1 \le b \le d+1)\chi(c \le d)\binom{2\delta-b}{d-b+1}\binom{d+b-1}{d}\binom{2\delta-1-c}{d-c}\binom{d+c}{d} \\ -\chi(b \le d)\chi(1 \le c \le d+1)\binom{2\delta-1-b}{d-b}\binom{d+b}{d}\binom{2\delta-c}{d-c+1}\binom{d+c-1}{d}. \end{cases}$$

If $b=c=0$ and $0 \le d \le \max\{b,c,1\}-1$, then $\chi(1 \le b) = \chi(1 \le c) = 0$ and $d=0$. If $b, 1 \le c$ and $d \le \max\{b,c,1\}-1$, then $\chi(c \le d) = 0$ and $\chi(c \le d+1)$ is non-zero only when $d = c-1$; so,

$$G_{b,c,\delta} = \begin{cases} \frac{\binom{3\delta-1}{2\delta}\binom{2\delta-1}{\delta}}{\delta}, & \text{if } b=c=0, \\ -\frac{\binom{3\delta-1}{2\delta+c-1}\binom{2\delta-c}{\delta}(2c-1)\chi(b\le c-1)}{\delta\binom{\delta-1}{c-1}^2}\binom{2\delta-1-b}{c-1-b}\binom{c-1+b}{c-1}\binom{2c-2}{c-1}, & \text{if } b, 1 \le c. \end{cases}$$

Use (7.10) to see that if $b=c=0$, then

$$(**)_{0,0,\delta} = \frac{4\binom{3\delta-1}{2\delta}\binom{2\delta-2}{\delta-1}}{(2\delta-2)^2\delta^2}\binom{2\delta-1}{2\delta-3}^2 + \frac{\binom{3\delta-1}{2\delta}\binom{2\delta-1}{\delta}}{\delta}.$$

Apply

$$\frac{4\binom{2\delta-1}{2\delta-3}^2}{(2\delta-2)^2} = (2\delta-1)^2 \quad \text{and} \quad \binom{2\delta-2}{\delta-1}(2\delta-1) = \binom{2\delta-1}{\delta}\delta$$

to see that

$$(**)_{0,0,\delta} = \frac{\binom{3\delta-1}{2\delta}\binom{2\delta-1}{\delta}}{\delta}((2\delta-1)+1) = \binom{3\delta-1}{2\delta}\binom{2\delta}{\delta},$$

as required in (7.5).

If $b, 1 \le c$, then (7.10) gives that

$$(**)_{b,c,\delta} = \begin{cases} \frac{\binom{3\delta-1}{2\delta+c-1}\binom{2\delta-1-c}{\delta-1}}{(2\delta-1-c)^2\binom{\delta}{c}^2}\frac{(c+1-b)(c-b)}{2(c+b)}\binom{2\delta-1-b}{2\delta-2-c}\binom{c+b}{c}\binom{2\delta-1-c}{2\delta-2-c}\binom{2c}{c} \\ -\frac{\binom{3\delta-1}{2\delta+c-1}\binom{2\delta-c}{\delta}(2c-1)\chi(b\le c-1)}{\delta\binom{\delta-1}{c-1}^2}\binom{2\delta-1-b}{c-1-b}\binom{c-1+b}{c-1}\binom{2c-2}{c-1} \end{cases}$$

In the present calculation, b and c are integers with $b \le c$; thus,

$$\chi(b \le c-1) = \begin{cases} 0 & \text{if } b=c \\ 1 & \text{if } b<c \end{cases}$$

and $(c-b)=(c-b)\chi(b\le c-1)$. It follows that

$$(**)_{b,c,\delta}=\chi(b\le c-1)\tbinom{3\delta-1}{2\delta+c-1}\begin{cases}\frac{\binom{2\delta-1-c}{\delta-1}}{(2\delta-1-c)^2\binom{\delta}{c}^2}\frac{(c+1-b)(c-b)}{2(c+b)}\binom{2\delta-1-b}{2\delta-2-c}\binom{c+b}{c}\binom{2\delta-1-c}{2\delta-2-c}\binom{2c}{c}\\ -\frac{\binom{2\delta-c}{\delta}(2c-1)}{\delta\binom{\delta-1}{c-1}^2}\binom{2\delta-1-b}{c-1-b}\binom{c-1+b}{c-1}\binom{2c-2}{c-1}.\end{cases}$$

Use

$$(2c-1)\tbinom{2c-2}{c-1}=\tbinom{2c}{c}\tfrac{c}{2},\quad \frac{\binom{2\delta-1-c}{2\delta-2-c}}{2\delta-1-c}=1,\quad \frac{(c+1-b)(c-b)}{(2\delta-c-1)}\tbinom{2\delta-1-b}{2\delta-2-c}=\tbinom{2\delta-1-b}{c-b-1}(2\delta-c),$$

$$\frac{\binom{c+b}{c}}{c+b}=\tbinom{c+b-1}{c-1}\tfrac{1}{c},\quad (2\delta-c)\tbinom{2\delta-1-c}{\delta-1}=\tbinom{2\delta-c}{\delta}\delta,\quad\text{and}\quad \delta\tbinom{\delta-1}{c-1}=\tbinom{\delta}{c}c.$$

to see that $(**)_{b,c,\delta}=0$ for $1,b\le c$, as required by (7.5), and the proof is complete. □

Lemma 7.13. *If δ is a positive integer, then the polynomial*

$$P_{2\delta}(A,B,C)+3AP_{2\delta-1}(A,B,C)+(-1)^{\delta}A(A+B+C)\operatorname{Poly}_{\delta-1,\delta,\delta}(A,B)\operatorname{Poly}_{\delta-1,\delta,\delta}(A,C)$$

of $\mathbb{Z}[A,B,C]$ is the zero polynomial.

Proof. Let X be the polynomial which is recorded in the statement. Use the fact that $3\binom{3\delta-1}{\delta-1}=\binom{3\delta}{\delta}$ to see that

$$X=\begin{cases}(-1)^{\delta+1}\operatorname{Poly}_{\delta,\delta,\delta}(A,B)\operatorname{Poly}_{\delta,\delta,\delta}(A,C)\\ +B\operatorname{Poly}_{\delta-1,\delta,\delta}(B,A)\operatorname{Poly}_{\delta,\delta,\delta}(A,C)\\ +C\operatorname{Poly}_{\delta,\delta,\delta}(A,B)\operatorname{Poly}_{\delta-1,\delta,\delta}(C,A)\\ \\ (-1)^{\delta}3A^2\operatorname{Poly}_{\delta-1,\delta,\delta-1}(A,B)\operatorname{Poly}_{\delta-1,\delta,\delta-1}(A,C)\\ +3AB\operatorname{Poly}_{\delta-1,\delta,\delta-1}(B,A)\operatorname{Poly}_{\delta-1,\delta,\delta-1}(A,C)\\ +3AC\operatorname{Poly}_{\delta-1,\delta,\delta-1}(A,B)\operatorname{Poly}_{\delta-1,\delta,\delta-1}(C,A)\\ \\ +(-1)^{\delta}A(A+B+C)\operatorname{Poly}_{\delta-1,\delta,\delta}(A,B)\operatorname{Poly}_{\delta-1,\delta,\delta}(A,C).\end{cases}$$

The polynomial X in $\mathbb{Z}[A,B,C]$ is homogeneous of degree 2δ. For non-negative integers b and c, the coefficient of $A^{2\delta-b-c}B^bC^c$ in X is

$$X_{b,c}=(-1)^{\delta+b+c}\begin{cases} -\binom{2\delta-b}{\delta}\binom{\delta+b}{\delta}\binom{2\delta-c}{\delta}\binom{\delta+c}{\delta}\\ +\binom{\delta+b-1}{\delta}\binom{2\delta-b}{\delta}\binom{2\delta-c}{\delta}\binom{\delta+c}{\delta}\\ +\binom{2\delta-b}{\delta}\binom{\delta+b}{\delta}\binom{\delta+c-1}{\delta}\binom{2\delta-c}{\delta}\\ \\ +3\binom{2\delta-b-1}{\delta}\binom{\delta+b-1}{\delta-1}\binom{2\delta-c-1}{\delta}\binom{\delta+c-1}{\delta-1}\\ +3\binom{\delta+b-1}{\delta}\binom{2\delta-b-1}{\delta-1}\binom{2\delta-c-1}{\delta}\binom{\delta+c-1}{\delta-1}\\ +3\binom{2\delta-b-1}{\delta}\binom{\delta+b-1}{\delta-1}\binom{\delta+c-1}{\delta}\binom{2\delta-c-1}{\delta-1}\\ \\ +\binom{2\delta-b-1}{\delta}\binom{\delta+b}{\delta}\binom{2\delta-c-1}{\delta}\binom{\delta+c}{\delta}\\ -\binom{2\delta-b}{\delta}\binom{\delta+b-1}{\delta}\binom{2\delta-c-1}{\delta}\binom{\delta+c}{\delta}\\ -\binom{2\delta-b-1}{\delta}\binom{\delta+b}{\delta}\binom{2\delta-c}{\delta}\binom{\delta+c-1}{\delta}. \end{cases}$$

We use

$$\begin{array}{rclcrcl} (\delta+b)\binom{\delta+b-1}{\delta-1} &=& \binom{\delta+b}{\delta}\delta & \quad & (2\delta-b)\binom{2\delta-1-b}{\delta} &=& \binom{2\delta-b}{\delta}(\delta-b)\\ (\delta+b)\binom{\delta+b-1}{\delta} &=& \binom{\delta+b}{\delta}b & & (2\delta-b)\binom{2\delta-1-b}{\delta-1} &=& \binom{2\delta-b}{\delta}\delta\\ (\delta+c)\binom{\delta+c-1}{\delta-1} &=& \binom{\delta+c}{\delta}\delta & & (2\delta-c)\binom{2\delta-1-c}{\delta} &=& \binom{2\delta-c}{\delta}(\delta-c)\\ (\delta+c)\binom{\delta+c-1}{\delta} &=& \binom{\delta+c}{\delta}c & & (2\delta-c)\binom{2\delta-1-c}{\delta-1} &=& \binom{2\delta-c}{\delta}\delta \end{array}$$

to write

$$(\delta+b)(\delta+c)(2\delta-b)(2\delta-c)X_{b,c}=(-1)^{\delta+b+c}X'_{b,c}\binom{2\delta-b}{\delta}\binom{\delta+b}{\delta}\binom{\delta+c}{\delta}\binom{2\delta-c}{\delta},$$

where $X'_{b,c}=X^{(1)}_{b,c}+X^{(2)}_{b,c}+X^{(3)}_{b,c}$, for

$$\begin{aligned} X^{(1)}_{b,c}&=-(2\delta-b)(\delta+b)(2\delta-c)(\delta+c)+(2\delta-b)b(2\delta-c)(\delta+c)+(2\delta-b)(\delta+b)(2\delta-c)c\\ &=-(2\delta-b)(2\delta-c)(\delta^2-bc),\\ X^{(2)}_{b,c}&=3\delta(\delta-b)\delta(\delta-c)+3\delta b\delta(\delta-c)+3\delta(\delta-b)\delta c=3\delta^2(\delta^2-bc),\\ X^{(3)}_{b,c}&=(\delta-b)(\delta+b)(\delta-c)(\delta+c)-(2\delta-b)b(\delta-c)(\delta+c)-(\delta-b)(\delta+b)(2\delta-c)c\\ &=(\delta^2-2\delta b-2\delta c+bc)(\delta^2-bc). \end{aligned}$$

Thus, $X'_{b,c}$ is equal to

$$[-(2\delta-b)(2\delta-c)+3\delta^2+(\delta^2-2\delta b-2\delta c+bc)](\delta^2-bc)=0.$$

The coefficient $X_{b,c}$ is also zero and $X=0$. $\square$

Section 8. Applications to Frobenius powers.

In this section the characteristic of the field $\boldsymbol{k}$ is $p > 0$. Keep in mind that the t^{th}-Frobenius power of $Q_{\boldsymbol{k},n,N}$ is

$$F^t(Q_{\boldsymbol{k},n,N}) = Q_{\boldsymbol{k},n,p^tN}.$$

Three questions are investigated in this section. When is there a module $Q_{\boldsymbol{k},n,N}$ of infinite projective dimension such that some Frobenius power of it has finite projective dimension? Is the tail of the resolution of $Q_{\boldsymbol{k},n,p^tN}$ (up to shift) eventually a periodic function of t? Can one use socle degrees to predict that the tail of the resolution of $Q_{\boldsymbol{k},n,p^tN}$ is a shift of the tail of the resolution of $Q_{\boldsymbol{k},n,N}$? The third question was answered in [8] in the case that both modules have finite projective dimension (hence the infinite tail of both resolutions is zero). It is shown in [7] how the socle degrees can be used to predict that the tail of the resolution of $Q_{\boldsymbol{k},n,p^tN}$ is a shift of the tail of the resolution of $Q_{\boldsymbol{k},n,N}$, **as a graded module**. We show that the condition of [7] actually produces an isomorphism of complexes **with differential**.

Subsection 8.1. When is there a module $Q_{\boldsymbol{k},n,N}$ of infinite projective dimension such that some Frobenius power of it has finite projective dimension?

At the Georgia State University – University of South Carolina Commutative Algebra Seminar in October, 2006, Florian Enescu asked, "When does the ring R have an ideal J so that R/J has infinite projective dimension but $R/J^{[p]}$ has finite projective dimension?" The participants at the seminar produced a partial answer to this result. Recall that the ideal I is *not Frobenius closed* if there is an element $r \in R$ with $r \notin I$, but $r^q \in I^{[q]}$ for some q. The ring $(R, \mathfrak{m})$ is *F-injective* if the Frobenius map induces an injective map on local cohomology modules $\mathrm{H}^i_{\mathfrak{m}}(R) \to \mathrm{H}^i_{\mathfrak{m}}(R)$, for every $0 \le i \le \dim(R)$. A local Cohen-Macaulay ring is F-injective if and only if all of its parameter ideals are Frobenius closed.

Proposition 8.1. ([Enescu, Kustin, Vraciu, Yao]) *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay ring. If R is not F-injective, then there exists an $\mathfrak{m}$-primary ideal J in R with $\operatorname{pd}_R R/J = \infty$, but $\operatorname{pd}_R R/J^{[p]}$ finite.*

Proof. The ring R is not F-injective; so, there exists a system of parameters $\boldsymbol{x} = (x_1, \dots x_d)$ for R (with $d = \dim R$) and an element u of R with $u \notin (\boldsymbol{x})$ but $u^p \in (\boldsymbol{x}^{[p]})$. Replace u by some multiple of u, if necessary, in order to have u in the ideal $(\boldsymbol{x}) : \mathfrak{m}$. The ideal $I = (\boldsymbol{x}, u)$ is linked to the maximal ideal of a local ring which is not regular; so, I has infinite projective dimension. On the other hand,

$I^{[p]} = (x_1^p, \dots x_d^p)$, because u^p is already in the ideal on the right side; thus, $I^{[p]}$ is generated by a regular sequence and has finite projective dimension. $\square$

It is natural to ask the following question.

Question 8.2. *Does the converse of Proposition 8.1 hold?*

Recently, Hailong Dao showed us that the answer to Question 8.2 would be "No" if the requirement that J is $\mathfrak{m}$-primary were removed.

Example 8.3. ([Dao]) Let $\boldsymbol{k}$ be a field of characteristic 2. If J is the ideal (x, z) of the ring $R = \boldsymbol{k}[x, y, z]/(xy - z^2)$, then I has infinite projective dimension; but $I^{[2]} = (x^2, z^2) = (x^2, xy) \cong (x, y)$ has finite projective dimension. The ring R is F-injective because $xy - z^2$ is not in $\mathfrak{M}^{[2]}$ where $\mathfrak{M}$ is the maximal ideal (x, y, z) of the polynomial ring $\boldsymbol{k}[x, y, z]$; see [4, Prop. 2.1 and Lemma 3.3].

It is known, by work of Fedder, which of the rings $R_{\boldsymbol{k},n}$ are not F-injective. Theorem 8.4 shows that the ideals (x^N, y^N, z^N) in the rings $R_{\boldsymbol{k},n}$ can not provide counter examples to Question 8.2.

Theorem 8.4. *Let $\boldsymbol{k}$ be a field of positive characteristic p and let $n \geq 2$ be an integer not divisible by p. The following statements are equivalent.*

(1) *There exists a positive integer N such that $\operatorname{pd}_{R_{\boldsymbol{k},n}} Q_{\boldsymbol{k},n,N}$ is infinite and $\operatorname{pd}_{R_{\boldsymbol{k},n}} Q_{\boldsymbol{k},n,pN}$ is finite.*
(2) *The integer n satisfies $n \geq 4$, or else $n = 3$ and $p \equiv 2 \mod 3$.*
(3) *The ring $R_{\boldsymbol{k},n}$ is not F-injective.*

Remark. The implication (3) $\Rightarrow$ (1) says more than Proposition 8.1 because Proposition 8.1 guarantees that when R is not F-injective, then there exists an $\mathfrak{m}$-primary ideal J with infinite projective dimension such that $J^{[p]}$ has finite projective dimension and the implication (3) $\Rightarrow$ (1) says that if, in addition, R has the form $R_{\boldsymbol{k},n}$, then J may be taken to have the form (x^N, y^N, z^N) for some N.

Proof. The equivalence of (2) and (3) is known from [4, Prop. 2.1 and Lemma 3.3]. The implication (1) $\Rightarrow$ (2) is established in Propositions 8.5 and 8.6 where it is shown that if $n = 2$ and p is odd or else if $n = 3$ and $p \equiv 1 \mod 3$, then

$$\operatorname{pd}_{R_{\boldsymbol{k},n}} Q_{\boldsymbol{k},n,pN} < \infty \implies \operatorname{pd}_{R_{\boldsymbol{k},n}} Q_{\boldsymbol{k},n,N} < \infty.$$

We do not consider $p = n = 3$ or $p = n = 2$.

We now prove (2) $\Rightarrow$ (1). If $p = 2$, $n \geq 3$, and $N = n - 1$, then $N < n < 2N$; and therefore Theorem 6.3 shows that $Q_{\boldsymbol{k},n,N}$ has infinite projective dimension, but $Q_{\boldsymbol{k},n,2N}$ has finite projective dimension.

Suppose $4 \le n$ and $p = 3$. Let $N = n+1$. Thus, $N = \theta n + r$ with $\theta = r = 1$. We see from Example 1.4 that $1 \in S_3$; so Theorem 6.2 shows that $Q_{\boldsymbol{k},n,N}$ has infinite projective dimension. On the other hand, $3N = \theta' n + r'$ with $\theta' = r' = 3$. (We still have $1 \le r' \le n-1$.) We see in Example 1.15 that $3 \in T_3$; so $3 \notin S_3$ and $Q_{\boldsymbol{k},n,3N}$ has finite projective dimension.

Suppose $4 \le n$ and $p \equiv 1 \mod 3$. Let $N = n(p - \frac{p-1}{3}) - 1$. We see that N is also equal to $\theta n + r$ for $\theta = p - \frac{p-1}{3} - 1$ and $r = n-1$. In the language of Definition 1.1, we have

$$\theta = \tfrac{2(p-1)}{3} = 2\pi_p \in S_p.$$

It follows from Theorem 6.2 that $Q_{\boldsymbol{k},n,N}$ has infinite projective dimension. On the other hand, $\frac{pN}{n} = p^2 - p\left(\frac{p-1}{3}\right) - \frac{p}{n}$, and

$$\left|p^2 - \tfrac{pN}{n}\right| = \tfrac{p^2-p}{3} + \tfrac{p}{n} < \tfrac{p^2-1}{3}.$$

The inequality on the right holds because

$$-\tfrac{p}{3} + \tfrac{p}{n} < -\tfrac{1}{3} \iff \tfrac{1}{3} + \tfrac{p}{n} < \tfrac{p}{3} \iff 1 < \left(\tfrac{n-3}{n}\right)p.$$

The parameters n and p satisfy $4 \le n$ (so $\frac{1}{4} \le \frac{n-3}{n}$) and $7 \le p$ (so $1 < \left(\frac{n-3}{n}\right)p$). Apply Theorem 6.3 with $J = 1$ and $q = p^2$ to conclude that $Q_{\boldsymbol{k},n,pN}$ has finite projective dimension.

Finally, suppose that p is an odd prime with $p \equiv 2 \mod 3$ and $n \ge 3$. Let $N = n(p - \frac{p+1}{3}) - 1$. We see that N is also equal to $\theta n + r$ for $\theta = p - \frac{p+1}{3} - 1$ and $r = n-1$. In the language of Definition 1.1, we have

$$\theta = \tfrac{2(p-2)}{3} = 2\pi_p \in S_p.$$

It follows from Theorem 6.2 that $Q_{\boldsymbol{k},n,N}$ has infinite projective dimension. On the other hand, $\frac{pN}{n} = p(p - \frac{p+1}{3}) - \frac{p}{n}$. Take J to be the odd integer $p - \frac{p+1}{3}$. We see that

$$|Jp - \tfrac{pN}{n}| = \tfrac{p}{n} < \tfrac{p+1}{3}.$$

The last inequality holds because $3 \le n$. It follows from Theorem 6.3 that $Q_{\boldsymbol{k},n,pN}$ has finite projective dimension. $\square$

Proposition 8.5. *Consider the data* $(\boldsymbol{k}, n, N)$ *with* $\boldsymbol{k}$ *a field of characteristic* p. *If* $p \equiv 1 \mod 3$ *and* $n = 3$, *then*

$$\operatorname{pd} Q_{\boldsymbol{k},n,pN} < \infty \implies \operatorname{pd} Q_{\boldsymbol{k},n,N} < \infty.$$

Proof. If n divides pN, then n divides N (since $n = 3$ and p is not divisible by 3) and $\operatorname{pd} Q_{\boldsymbol{k},n,N}$ is automatically finite. Throughout the rest of the proof we assume that n does not divide pN. According to Theorem 6.3, there exists an odd integer J and a power $q = p^e$ of p with $e \geq 1$ and $|Jq - \frac{pN}{3}| < \frac{q-1}{3}$. Multiply by 3 to see $|3Jq - pN| < q - 1$. The integer pN is divisible by p. There are no integers in the interval $(q-p, q-1)$ which are divisible by p; hence, $|3Jq - pN| \leq q - p$. The integers $q - p$ and $3Jq$ are divisible by 3. The integer pN is not divisible by 3; so $|3Jq - pN| < q - p$. Let $q' = \frac{q}{p}$ and divide by $3p$ to see that $|Jq' - \frac{N}{3}| < \frac{q'-1}{3}$. This inequality shows that $1 < q'$ and Theorem 6.3 gives $\operatorname{pd} Q_{\boldsymbol{k},n,N} < \infty$. □

Proposition 8.6. *Consider the data $(\boldsymbol{k}, n, N)$ with $\boldsymbol{k}$ a field of characteristic p. If p is an odd prime and $n = 2$, then*

$$\operatorname{pd} Q_{\boldsymbol{k},n,pN} < \infty \implies \operatorname{pd} Q_{\boldsymbol{k},n,N} < \infty.$$

Proof. If n divides pN, then n divides N (since $n = 2$ and p is not divisible by 2) and $\operatorname{pd} Q_{\boldsymbol{k},n,N}$ is automatically finite. Throughout the rest of the proof we assume that n does not divide pN. According to Theorem 6.3, there exists an odd integer J and a power $q = p^e$ of p with $e \geq 1$ and

$$|Jq - \tfrac{pN}{2}| < \{\tfrac{q}{3}\}. \tag{8.7}$$

We first treat the case $p = 3$. In this case, $\{\frac{q}{3}\} = \frac{q}{3}$. Divide (8.7) by 3 to obtain $|J\frac{q}{3} - \frac{N}{2}| < \frac{q}{9}$. The hypotheses that N is an odd integer guarantees that $q > 3$ and therefore Theorem 6.3 shows that $\operatorname{pd} Q_{\boldsymbol{k},n,N}$ is finite.

Henceforth, we assume that $p \geq 5$. Multiply (8.7) by 6 to see

$$|6Jq - 3pN| < 6\{\tfrac{q}{3}\}. \tag{8.8}$$

The integer $3pN$ is divisible by $3p$. The intervals

$$\begin{cases} (2(q-p), 6\{\frac{q}{3}\}) & \text{if } \frac{q}{p} \equiv 1 \mod 3 \\ (6\{\frac{q}{3}\}, 2(q+p)) & \text{if } \frac{q}{p} \equiv 2 \mod 3 \end{cases}$$

do not contain any integers which are divisible by $3p$. Hence, (8.8) is equivalent to

$$\begin{cases} |6Jq - 3pN| \leq 2(q-p) & \text{if } \frac{q}{p} \equiv 1 \mod 3 \\ |6Jq - 3pN| < 2(q+p) & \text{if } \frac{q}{p} \equiv 2 \mod 3. \end{cases}$$

The integers $6Jq$ and $2(q-p)$ are divisible by 2. The integer $3pN$ is not divisible by 2; so

$$\begin{cases} |6Jq - 3pN| < 2(q-p) & \text{if } \frac{q}{p} \equiv 1 \mod 3 \\ |6Jq - 3pN| < 2(q+p) & \text{if } \frac{q}{p} \equiv 2 \mod 3. \end{cases}$$

Let $q' = \frac{q}{p}$ and divide by $6p$ to see that

$$\begin{cases} |Jq' - \frac{N}{2}| < \frac{q'-1}{3} & \text{if } \frac{q}{p} \equiv 1 \mod 3 \\ |Jq' - \frac{N}{2}| < \frac{q'+1}{3} & \text{if } \frac{q}{p} \equiv 2 \mod 3. \end{cases}$$

Thus, $1 < q'$, $|Jq' - \frac{N}{2}| < \{\frac{q'}{3}\}$, and Theorem 6.3 gives $\operatorname{pd} Q_{\boldsymbol{k},n,N} < \infty$. □

Subsection 8.2. Is the tail of the resolution of $Q_{\boldsymbol{k},n,p^tN}$ (up to shift) eventually a periodic function of t?

Data 8.9. Fix data $(\boldsymbol{k}, n)$, where $\boldsymbol{k}$ is a field of characteristic $p > 0$ and $n \geq 2$ is an integer which is relatively prime to p. Fix the ring $R = R_{\boldsymbol{k},n}$ and let Q_N denote the R-module $Q_{\boldsymbol{k},n,N}$. Let $\boldsymbol{o}$ be the the order of p in the group of multiplicative units in the ring $\mathbb{Z}/(n)$.

In Theorem 8.10, we show that, for each positive integer N, the tail of the resolution of Q_{p^tN}, up to shift, is eventually a periodic function of t of period at most $2\boldsymbol{o}$. The "tail of the resolution" of Q_N is equal to the resolution of the second syzygy of the R-module Q_N, which we denote by $\operatorname{syz}_2 Q_N$ (see (1.25)); therefore, the tail of the resolution of Q_{qN} is isomorphic to a shift of the tail of the resolution of Q_N if and only if $\operatorname{syz}_2 Q_{qN}$ is isomorphic to a shift of $\operatorname{syz}_2 Q_N$.

Theorem 8.10. *Adopt Data 8.9. Then there exists a power $q = p^e$, of p, for some integer $e \geq 1$, such that, for all positive integers N, there exists an integer $t_0(N)$, which depends on N, so that*

$$\operatorname{syz}_2 Q_{q^sp^tN} \textit{ is isomorphic to a shift of } \operatorname{syz}_2 Q_{p^tN} \tag{8.11}$$

for all integers s and t with $s \geq 0$ and $t \geq t_0(N)$. Furthermore, e may be taken to be at most $2\boldsymbol{o}$.

Proof. There are two cases. Either

Case 1: there exists an integer $t_0 = t_0(N)$ (which depends on N) such that $Q_{p^{t_0}N}$ has finite projective dimension; or else,

Case 2: $\operatorname{pd} Q_{p^tN}$ is infinite for all t.

In the first case, take $q = p$. It follows from the Theorem of Peskine and Szpiro [11, Thm. I.7] that $Q_{q^s p^t N}$ has finite projective dimension; and therefore $\operatorname{syz}_2 Q_{q^s p^t N}$ is a free R-module, for all $s \geq 0$ and $t \geq t_0$. Thus, (8.11) holds.

Recall from Theorem 6.3 that if $p = 2$, then Q_N has finite projective dimension for all $N \geq n$. Thus, $p = 2$ is covered in the first case.

Henceforth, we focus on the second case with $p \geq 3$. Take $t_0(N)$ to be zero. We first identify an exponent e so that $q = p^e$ has the form $bn + 1$ for some positive even integer b. Let $q' = p^{\boldsymbol{o}}$ and write $q' = b'n + 1$. If b' is even, then take $e = \boldsymbol{o}$, $q = q'$ and $b = b'$. If b' is odd, then n must be even since $b'n + 1$ is the odd integer q'. Observe that $(q')^2 = ((b')^2 n + 2b')n + 1$, with $b = (b')^2 n + 2b'$ even. In this case, take $e = 2\boldsymbol{o}$ and $q = p^e$.

Take any $t \geq 0$. Write $p^t N = an + r$, with $1 \leq r \leq n - 1$. (The remainder r can not be zero because $\operatorname{pd} Q_{p^t N} = \infty$.) It follows that $qp^t N = (bp^t N + a)n + r$. Observe that a and $bp^t N + a$ have the same parity and the remainder r is the same in both numbers $p^t N$ and $qp^t N$. The modules $Q_{p^t N}$ and $Q_{qp^t N}$ both have infinite projective dimension; so the integers a and $bp^t N + a$ are in S_p according Theorem 6.2. Corollary 3.7 shows that $\operatorname{syz}_2 Q_{\boldsymbol{k},n,qp^t N}$ and $\operatorname{syz}_2 Q_{\boldsymbol{k},n,p^t N}$ are both isomorphic to a shift of

$$\begin{cases} \operatorname{coker} \varphi_{r,n-r} & \text{if } a \text{ is odd} \\ \operatorname{coker} \varphi_{n-r,r} & \text{if } a \text{ is even.} \end{cases}$$

One may iterate this procedure to see that (8.11) holds. $\square$

Given Data 8.9, it is natural to ask for a bound t_0, which depends only on p and n, such that, if $N \geq 1$ is an integer, then

$$\operatorname{pd} Q_{p^t N} = \infty \text{ for some } t \geq t_0 \implies \operatorname{pd} Q_{p^t N} = \infty \text{ for all } t \geq 0. \tag{8.12}$$

We have some partial results in the direction of describing such a number t_0.

- If $p = 2$, then $\operatorname{pd} Q_{p^t N} < \infty$ for all $t \geq n$; so, any $t_0 \geq \log_2 n$ has property (8.12) for vacuous reasons.
- If $n = 2$ and p is odd, or $n = 3$ and $p \equiv 1 \mod 3$, then Propositions 8.5 and 8.6 show that $t_0 = 0$ has property (8.12).
- If $n = 3$ and $p \equiv 2 \mod 3$ or $n \geq 4$ and p is odd, then Proposition 8.15 shows that $t_0 = 2\boldsymbol{o}$ satisfies (8.12) for all $N \equiv 1 \mod n$. This special case of (8.12) includes the most important case $N = 1$, where $Q_N = \boldsymbol{k}$: if $\operatorname{pd} Q_{p^{2\boldsymbol{o}}} = \infty$, then $\operatorname{pd} Q_{p^t} = \infty$ for all t.

Lemma 8.13. *Let p be an odd prime integer, and n be an integer with either $n \geq 4$ or else $n = 3$ and $p \equiv 2 \mod 3$. Suppose $q = p^e$ and $q = bn + 1$ for some positive integers b and e. If a is a non-negative integer with $qa + b$ in S_p, then a is even.*

Proof. Assume a is odd. We prove that $qa+b$ is in T_p. We treat two cases. In the first case either $n \geq 4$ or $n=3$ and $q \equiv 2$. In the second case $n=3$ and $q \equiv 1$. In the first case we have

$$|(qa+b)-qa| = b = \tfrac{q-1}{n} < \{\tfrac{q}{3}\}$$

and Remark 1.12 shows that $qa+b$ is in T_p. We justify the inequality $\frac{q-1}{n} < \{\frac{q}{3}\}$. If $4 \leq n$, then $\frac{q-1}{n} < \frac{q-1}{3} \leq \{\frac{q}{3}\}$. If $n=3$ and $q \equiv 2$, $\frac{q-1}{n} = \frac{q-1}{3} < \frac{q+1}{3} = \{\frac{q}{3}\}$.

Now we consider the case where $n=3$, $p \equiv 2 \mod 3$, and $q \equiv 1 \mod 3$. Let $q' = q/p$. Observe that $q' \equiv 2 \mod 3$. We see that

$$qa+b = qa + \tfrac{q-1}{3} = qa + \tfrac{pq'-1}{3} = qa + \tfrac{(p+1)q'-(q'+1)}{3} = q'(pa + \tfrac{p+1}{3}) - \tfrac{q'+1}{3}.$$

The integer $pa + \frac{p+1}{3}$ is odd; so, $qa+b$ is in T_p by Definition 1.10. $\square$

Observation 8.14. *Suppose $p \geq 5$ is a prime integer, $n \geq 3$ is an integer, and $q = p^e$ for some integer $e \geq 1$. Suppose further that $q = bn+1$ and $b = \sum_{i=0}^{E} b_i p^i$ for integers b_i with $|b_i|$ at most $2\lfloor \frac{p}{3} \rfloor$ for all i. Then $b_e = \cdots = b_E = 0$.*

Proof. Write $b = A+B$, with $A = \sum_{i=0}^{e-1} b_i p^i$ and $B = \sum_{i=e}^{E} b_i p^i$. If $b_i \neq 0$ for some i with $e \leq i \leq E$, then $|B|$ is equal to p^e times a positive integer; thus, $p^e \leq |B|$. On the other hand, $|A| \leq 2\lfloor \frac{p}{3} \rfloor \frac{p^e-1}{p-1}$; so,

$$q \leq |B| = |(A+B)-A| \leq |A+B|+|A| \leq \tfrac{q-1}{n} + 2\left\lfloor \tfrac{p}{3} \right\rfloor \tfrac{q-1}{p-1} \leq (q-1)\left(\tfrac{1}{n} + \tfrac{2}{3}\right) \leq q-1.$$

We used the fact that $A+B = b = \frac{q-1}{n}$. We also used the fact that 3 does not divide p, so $\lfloor \frac{p}{3} \rfloor \leq \frac{p-1}{3}$. This contradiction shows that $b_i = 0$ for $e \leq i \leq E$. $\square$

Proposition 8.15. *Adopt Data 8.9 with $p \geq 3$. Assume that $n \geq 4$ or else $n=3$ and $p \equiv 2 \mod 3$. Let q be the integer $p^{\boldsymbol{o}}$. Write $q = bn+1$ and $N = an+1$ for some integers b and a. The following statements are equivalent:*

(1) $\operatorname{pd} Q_{p^t N} = \infty$ *for all integers* $t \geq 0$,
(2) $\operatorname{pd} Q_{q^2 N} = \infty$,
(3) *the integers a and b are even elements of S_p.*

Furthermore, if the above equivalent conditions are in effect, then $\operatorname{syz}_2 Q_{q^s N}$ is isomorphic to a shift of $\operatorname{syz}_2 Q_N$, for all non-negative integers s.

Proof. Observe, by induction, that

$$\begin{aligned} q^0 N &= an+1 \\ q^1 N &= (qa+b)n+1 \\ q^2 N &= (q^2 a + qb + b)n + 1 \\ q^s N &= \left[q^s a + \left(\textstyle\sum_{i=0}^{s-1} q^i\right) b\right] n + 1. \end{aligned}$$

(2) $\Rightarrow$ (3). Theorem 6.2 shows that the integers a, $qa+b$, and $q^2a+qb+b$ all are in S_p. Apply Lemma 8.13 to a and to $qa+b$. We have $qa+b$ and $q(qa+b)+b$ are both in S_p; so we conclude that a and $qa+b$ are both even. It follows that b is even.

We next show that b is in S_p. We use the special p-adic expansion of b. We treat the cases $p=3$ and $p\ge 5$ separately. We first assume that $p=3$. Use Remark 1.7. The fact that a and $qa+b$ are both even elements of S_p ensures that a and $qa+b$ are divisible by 4. So, b is also divisible by 4. Write

$$a=4\sum_{i=0}^{r}\epsilon_i 3^i \quad\text{and}\quad b=4\sum_{i=0}^{s}\delta_i 3^i \tag{8.16}$$

for $\epsilon_i\in\{0,1\}$ and $\delta_i\in\{0,1,2\}$. Observe that

$$3^{s+1}\le 4(3^s)\le b=\frac{q-1}{3}<\frac{q}{3}=3^{\boldsymbol{o}-1};$$

so, $s\le \boldsymbol{o}-2$ and

$$qa+b=4(\epsilon_r 3^{r+\boldsymbol{o}}+\cdots+\epsilon_0 3^{\boldsymbol{o}}+\delta_s 3^s+\cdots+\delta_0). \tag{8.17}$$

The fact that $qa+b$ is in S_p says that every coefficient in the expansion (8.17) is in $\{0,1\}$. Thus, every $\delta_i\in\{0,1\}$ and $b\in S_p$.

Now we assume $p\ge 5$. Expand b and a in the p-adic expansion as described in Notation 1.5. Observation 8.14 shows that

$$b=\sum_{i=0}^{\boldsymbol{o}-1} b_i p^i.$$

The p-adic expansion of $qa+b$, in the sense of Notation 1.5, is obtained by concatenating the p-adic expansion of qa with the p-adic expansion of b. We have shown that $qa+b$ is an even element of S_p. Remark 1.6 shows that every coefficient of the p-adic expansion of $qa+b$, in the sense of Notation 1.5, is even. Therefore, every coefficient of b is even; and therefore Remark 1.6 shows that b are even elements of S_p. The proof of (2) $\Rightarrow$ (3) is complete.

The implication (3) $\Rightarrow$ (1) now follows readily. If $p=3$, then a and b are given in (8.16) with all coefficients ϵ_i and δ_i from the set $\{0,1\}$. If $p\ge 5$, every coefficient of the p-adic expansions of a and b, in the sense of Notation 1.5, is even. For all characteristics $p\ge 3$, the p-adic expansion of $\lfloor\frac{q^sN}{n}\rfloor$ is obtained by concatenating the p-adic expansions of $q^sa, q^{s-1}b,\dots qb, b$. The form of the resulting p-adic expansion shows that $\lfloor\frac{q^sN}{n}\rfloor\in S_p$; and therefore, Q_{q^sN} has infinite projective dimension by Theorem 3.5. The theorem of Peskine and Szpiro [11, Thm. I.7] ensures that Q_{p^tN} has infinite projective dimension for all non-negative integers t.

The final assertion is an immediate consequence of Theorem 3.5 where it is shown that $\operatorname{syz}_2 Q_{q^sN}$ is a shift of the cokernel of $\varphi_{n-1,1}$. $\square$

Subsection 8.3. Can one use socle degrees to predict that the tail of the resolution of $Q_{\boldsymbol{k},n,p^tN}$ is a shift of the tail of the resolution of $Q_{\boldsymbol{k},n,N}$?

We denote the socle of Q by $\operatorname{soc} Q$, see Definition 1.23; and, if $\mathbb{F}$ is a complex, then $\mathbb{F}_{\geq i}$ is the truncation $\cdots \to F_{i+1} \to F_i$ of $\mathbb{F}$, see (1.25).

Theorem 8.18. *Let $\boldsymbol{k}$ be a field and n, N_1, and N_2 be positive integers. Write R for the ring $R_{\boldsymbol{k},n}$ and Q_{N_i} for the R-module $Q_{\boldsymbol{k},n,N_i}$. Assume that Q_{N_1} and Q_{N_2} both have infinite projective dimension over R. Let $\mathbb{F}_{i,\bullet}$ be the minimal homogeneous resolution of Q_{N_i} by free R-modules and let $N_i = a_in + r_i$ with $1 \leq r_i \leq n-1$. The following statements are equivalent:*

1. $\operatorname{soc} Q_{N_2}$ *is isomorphic to a shift of* $\operatorname{soc} Q_{N_1}$ *as a graded vector space,*
2. $\mathbb{F}_{2,\geq 2}$ *is isomorphic to a shift of* $\mathbb{F}_{1,\geq 2}$ *as a graded R-**module**,*
3. $\mathbb{F}_{2,\geq 2}$ *is isomorphic to a shift of* $\mathbb{F}_{1,\geq 2}$ *as a graded **complex**,*
4. *the R-module* $\operatorname{syz}_2 Q_{N_2}$ *is isomorphic to a shift of* $\operatorname{syz}_2 Q_{N_1}$*, and*
5. *either* (a) $a_1 + a_2$ *is even and* $r_1 = r_2$*; or else,* (b) $a_1 + a_2$ *is odd and* $r_1 + r_2 = n$.

Furthermore, if $N_2 = qN_1$ for some positive integer q and conditions (1)–(5) *occur, then*

$$\operatorname{soc} Q_{N_2} \cong \operatorname{soc} Q_{N_1}(-w), \quad \operatorname{syz}_2 Q_{N_2} \cong \operatorname{syz}_2 Q_{N_1}(-w), \quad \textit{and} \quad \mathbb{F}_{2,\geq 2} \cong \mathbb{F}_{1,\geq 2}(-w)$$

for $w = \frac{3}{2}N_1(q-1)$.

Remark. The implication (1) $\Rightarrow$ (2) is [7, Thm. 1.1]. We have an explicit formula for the socle degrees of Q_{N_i} (see Theorem 3.5); so, we may verify hypothesis (c) of [7, Thm. 1.1] directly. The parameter $b + 2a$ from [7] is $3N_i + 2n - 6$ in the present notation. The techniques of [7] were not able to prove that (1) $\Rightarrow$ (3). One of the motivations for the present work was to establish this implication. The shift $\frac{3}{2}N_1(q-1)$ is identified in [7, Cor. 2.1].

Proof. Assertions (3) and (4) are equivalent because $\mathbb{F}_{i,\geq 2}$ is the minimal resolution of $\operatorname{syz}_2 Q_{N_i}$. It is clear that (3) $\Rightarrow$ (2) because (3) is a statement about graded Betti numbers and differentials; whereas (2) is only a statement about graded Betti numbers.

Apply Theorem 6.2 to see that $a_i \in S_c$, where c is the characteristic of $\boldsymbol{k}$, and apply Theorem 3.5 to read the socle degrees of Q_{N_i} and the resolution $\mathbb{F}_{i,\bullet}$ from the data $N_i = a_in + r_i$. The assertion (2) $\Rightarrow$ (1) may be read from Theorem 3.5 where it is shown that

$$\operatorname{soc} Q_{N_i} \cong \frac{\mathbb{F}_{i,3}}{(x,y,z)\mathbb{F}_{i,3}}(+3) \tag{8.19}$$

as a graded vector space.
(1) $\Rightarrow$ (5) The socle degrees of Q_{N_i} have the form $D_i\!:\!3, d_i\!:\!1$ for (D_i, d_i) given in Theorem 3.5. Condition (1) asserts that there exists an integer w with $D_2 = D_1+w$ and $d_2 = d_1 + w$. It follows that

$$D_2 - d_2 = D_1 - d_1. \tag{8.20}$$

We see from Theorem 3.5 that

$$D_i - d_i = \begin{cases} -n + 2r_i & \text{if } a_i \text{ is odd} \\ n - 2r_i & \text{if } a_i \text{ is even.} \end{cases}$$

If a_1 and a_2 have the same parity, then (8.20) shows that $r_1 = r_2$. If a_1 and a_2 have the different parity, then (8.20) shows that $n = r_1 + r_2$.
(5) $\Rightarrow$ (4) If hypothesis (a) holds, then Theorem 3.5 shows that $\operatorname{syz}_2 Q_{N_1}$ and $\operatorname{syz}_2 Q_{N_2}$ are both isomorphic to a shift of

$$\begin{cases} \operatorname{coker} \varphi_{n-r_1,r_1} & \text{if } a_1 \text{ is even} \\ \operatorname{coker} \varphi_{r_1,n-r_1} & \text{if } a_1 \text{ is odd.} \end{cases}$$

If hypothesis (b) holds with a_1 even, then

$$N_2 = (a_2 - 1)n + (n - r_2) = (a_2 - 1)n + r_1,$$

with $a_2 - 1$ even and Theorem 3.5 shows that $\operatorname{syz}_2 Q_{N_1}$ and $\operatorname{syz}_2 Q_{N_2}$ are both isomorphic to a shift of $\operatorname{coker} \varphi_{n-r_1,r_1}$.

Finally, we assume (1) – (5) hold. These conditions guarantee that there exist integers w_1, w_2, w_3 with

$$\operatorname{soc} Q_{N_2} \cong \operatorname{soc} Q_{N_1}(-w_1), \quad \operatorname{syz}_2 Q_{N_2} \cong \operatorname{syz}_2 Q_{N_1}(-w_2), \quad \text{and} \quad \mathbb{F}_{2,\geq 2} \cong \mathbb{F}_{1,\geq 2}(-w_3).$$

The complex $\mathbb{F}_{i,\geq 2}$ is a resolution of $\operatorname{syz}_2 Q_{N_i}$; so $w_2 = w_3$, and the isomorphism (8.19) shows that $w_1 = w_3$. We now identify the common value $w_1 = w_2 = w_3$, when $N_2 = qN_1$. Condition (5) is in effect. If a_1 and a_2 are both even, then Theorem 3.5 gives

$$w_1 = \tfrac{3}{2}(a_2 n - a_1 n) = \tfrac{3}{2}([a_2 n + r] - [a_1 n + r]) = \tfrac{3}{2}(qN_1 - N_1).$$

If a_1 and a_2 are both odd, then

$$w_1 = \tfrac{3}{2}([a_2 + 1]n - [a_1 + 1]n) = \tfrac{3}{2}([a_2 n + r] - [a_1 n + r]) = \tfrac{3}{2}(qN_1 - N_1).$$

If a_1 is even and a_2 is odd, then

$$w_1 = \tfrac{3}{2}(a_2 + 1)n - (\tfrac{3}{2}a_1 n + 3r) = \tfrac{3}{2}([(a_2 + 1)n - r] - [a_1 n + r]) = \tfrac{3}{2}(qN_1 - N_1). \quad \square$$

Section 9. Two variables.

Consider

$$\bar{R}_{\boldsymbol{k},n} = \boldsymbol{k}[x,y]/(x^n+y^n) \quad \text{and} \quad \bar{Q}_{\boldsymbol{k},n,N} = \bar{R}_{\boldsymbol{k},n}/(x^N,y^N), \tag{9.1}$$

where $\boldsymbol{k}$ is a field. The results in this section were announced in the Introduction to [7]. In particular, Example 9.8 exhibits a situation where the tail of the resolution of $F^t(\bar{Q}_{\boldsymbol{k},n,N})$, after shifting, is periodic as a function of t, with an arbitrarily large period. We have used F to represent the Frobenius functor.

Observation 9.2 and Corollary 9.3 are the two variable analogues of Theorem 3.5. Observation 9.4 is the two variable analogue of Theorem 8.18. Two phenomenon occur in 3 variables that do not occur in 2 variables. Let $N = an + r$, with $0 \le r \le n-1$. In 3 variables, a plays an important role; in 2 variables, a is irrelevant. In 2 variables $\operatorname{pd}_{\bar{R}_{\boldsymbol{k},n}} \bar{Q}_{\boldsymbol{k},n,N}$ is infinite whenever n does not divide N; in 3 variables $Q_{\boldsymbol{k},n,N}$ often has finite projective dimension over $R_{\boldsymbol{k},n}$.

Observation 9.2. *Let x, y be a regular sequence in a ring P. Let a, n, and r be fixed integers with $0 \le r \le n-1$ and $0 \le a$, and let $f = x^n + y^n$, and $N = an + r$. Then the ideal $K = (x^N, y^N, f)$ of P is perfect of grade two and P/K is resolved by*

$$0 \to P^2 \xrightarrow{d_2} P^3 \xrightarrow{d_1} P \to P/K \to 0,$$

with

$$d_1 = [\,x^N \quad y^N \quad f\,] \quad \text{and} \quad d_2 = \begin{bmatrix} x^{n-r} & (-1)^{a-1}y^r \\ (-1)^a y^{n-r} & x^r \\ L & M \end{bmatrix},$$

for

$$L = -\sum_{i=0}^{a} (x^n)^{a-i}(-y^n)^i \quad \text{and} \quad M = (-1)^a x^r y^r \sum_{i=0}^{a-1} (x^n)^{a-1-i}(-y^n)^i.$$

Proof. It suffices to show that the entries of d_1 are the signed maximal order minors of d_2. (See (5.2) if necessary.) Let Δ_i be $(-1)^{i+1}$ times the determinant of d_2 with row i deleted. We see that $\Delta_3 = f$,

$$\begin{aligned}
\Delta_2 &= -x^{n-r}M + (-1)^{a-1}y^r L \\
&= (-1)^{a+1}x^n y^r \sum_{i=0}^{a-1}(x^n)^{a-1-i}(-y^n)^i + (-1)^a y^r \sum_{i=0}^{a}(x^n)^{a-i}(-y^n)^i \\
&= (-1)^a y^r \left(-\sum_{i=0}^{a-1}(x^n)^{a-i}(-y^n)^i + \sum_{i=0}^{a}(x^n)^{a-i}(-y^n)^i \right) = (-1)^a y^r \left((-y^n)^a\right) \\
&= y^N,
\end{aligned}$$

and

$$\begin{aligned}\Delta_1 &= (-1)^a y^{n-r}M - x^r L\\ &= x^r y^n \sum_{i=0}^{a-1}(x^n)^{a-1-i}(-y^n)^i + x^r \sum_{i=0}^{a}(x^n)^{a-i}(-y^n)^i\\ &= x^r \left(-\sum_{i=0}^{a-1}(x^n)^{a-1-i}(-y^n)^{i+1} + \sum_{i=0}^{a}(x^n)^{a-i}(-y^n)^i\right)\\ &= x^r \left(-\sum_{i=1}^{a}(x^n)^{a-i}(-y^n)^{i} + \sum_{i=0}^{a}(x^n)^{a-i}(-y^n)^i\right)\\ &= x^N. \quad \square\end{aligned}$$

Corollary 9.3. *Keep the notation of Observation 9.2. Assume that f is a regular element in P. Let $R = P/(f)$,*

$$D = \begin{bmatrix} x^{n-r} & (-1)^{a-1}y^r \\ (-1)^a y^{n-r} & x^r \end{bmatrix}, \quad \text{and} \quad \check{D} = \begin{bmatrix} x^r & (-1)^a y^r \\ (-1)^{a+1}y^{n-r} & x^{n-r} \end{bmatrix}.$$

Then the R-resolution of $R/(x^N, y^N)R$ is

$$\dots \xrightarrow{\check{D}} R^2 \xrightarrow{D} R^2 \xrightarrow{\check{D}} R^2 \xrightarrow{D} R^2 \xrightarrow{[\,x^N \quad y^N\,]} R \to R/(x^N, y^N)R \to 0.$$

Proof. Notice that D is the top two rows of d_2 and that $\check{D}$ is the classical adjoint of D; so $D\check{D} = \check{D}D = fI_2$ in P. It is clear that the proposed resolution is a complex. If $Dv = fw$ in P, then multiply by $\check{D}$ to learn that $fv = f\check{D}w$; but f is a regular element in P; so $v = \check{D}w$. One may also reverse the roles of D and $\check{D}$. If $[\,x^N \quad y^N\,]\,v = af$ in P, then $\left[\begin{smallmatrix} v \\ -a \end{smallmatrix}\right]$ is in $\ker d_1 = \operatorname{im} d_2$; so there exists u in P^2 with

$$\begin{bmatrix} D \\ \hline L\ M \end{bmatrix} u = \begin{bmatrix} v \\ -a \end{bmatrix};$$

in particular, $Du = v$. $\square$

Observation 9.4. *Let $\boldsymbol{k}$ be a field and n, N_1, and N_2 be positive integers. Write R for the ring $\bar{R}_{\boldsymbol{k},n}$ and Q_{N_i} for the R-module $\bar{Q}_{\boldsymbol{k},n,N_i}$, as described in* (9.1). *Let $\mathbb{F}_{i,\bullet}$ be the minimal homogeneous resolution of Q_{N_i} by free R-modules. Then the following statements are equivalent:*

(1) $\operatorname{soc} Q_{N_2}$ *is isomorphic to a shift of* $\operatorname{soc} Q_{N_1}$ *as a graded vector space,*

(2) $\mathbb{F}_{2,\geq 1}$ *is isomorphic to a shift of* $\mathbb{F}_{1,\geq 1}$ *as a graded* R-**module**,
(3) $\mathbb{F}_{2,\geq 1}$ *is isomorphic to a shift of* $\mathbb{F}_{1,\geq 1}$ *as a graded* **complex**,
(4) *the* R*-module* $\operatorname{syz}_1 Q_{N_2}$ *is isomorphic to a shift of* $\operatorname{syz}_1 Q_{N_1}$, *and*
(5) $N_1 \equiv \pm N_2 \mod n$.

Furthermore, if $N_2 = qN_1$ *for some positive integer* q *and conditions* (1)–(5) *occur, then*

$$\operatorname{soc} Q_{N_2} \cong \operatorname{soc} Q_{N_1}(-w), \quad \operatorname{syz}_1 Q_{N_2} \cong \operatorname{syz}_1 Q_{N_1}(-w), \quad \textit{and} \quad \mathbb{F}_{2,\geq 1} \cong \mathbb{F}_{1,\geq 1}(-w)$$

for $w = N_1(q-1)$.

Proof. Let $N_i = a_i n + r_i$ with $1 \leq r_i \leq n-1$, f be the polynomial $x^n + y^n$ in $P = \boldsymbol{k}[x,y]$, and K_i be the ideal (x^{N_i}, y^{N_i}, f) of P. Observation 9.2 shows that the P-resolution of P/K_i is

$$0 \to P(-N_i - n + r_i) \oplus P(-N_i - r_i) \to P(-N_i)^2 \oplus P(-n) \to P.$$

It follows from Remark 1.24 that the socle degrees of $P/K_i = Q_{N_i}$ are

$$\{N_i + n - r_i - 2, N_i + r_i - 2\}. \tag{9.5}$$

We also know from Corollary 9.3 that the first syzygy module of the R-modules Q_{N_i} is presented by

$$\begin{matrix} R(-N_i - n + r_i) \\ \oplus \\ R(-N_i - r_i) \end{matrix} \xrightarrow{D_i} R(-N_i)^2 \to \operatorname{syz}_1^R(Q_{N_i}) \to 0, \tag{9.6}$$

with

$$D_i = \begin{bmatrix} x^{n-r_i} & -y^{r_i} \\ y^{n-r_i} & x^{r_i} \end{bmatrix}.$$

We read from (9.5) and (9.6) that

$$\operatorname{soc} Q_{N_i} \cong \frac{\mathbb{F}_{i,2}}{(x,y)\mathbb{F}_{i,2}}(+2) \tag{9.7}$$

as a graded vector space.

We have (4) $\iff$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1) exactly as in the proof of Theorem 8.18 once (8.19) is replaced with (9.7). The socle degrees of Q_{N_i} are $\{d_i, d_i'\}$, as given in (9.5). It follows that

$$\begin{aligned} (1) \implies |d_1 - d_1'| = |d_2 - d_2'| \iff |n - 2r_1| = |n - 2r_2| \\ \implies r_1 = r_2 \text{ or } r_1 + r_2 = n \iff N_1 \equiv \pm N_2 \mod n \iff (5). \end{aligned}$$

(5) $\implies$ (4) The R-module $\operatorname{syz}_1(Q_{N_i})$ is presented by D_i as shown in (9.6). We see that

$$D_2 = \begin{cases} D_1 & \text{if } r_1 = r_2 \\ J^{\mathrm{T}} D_1 J & \text{if } r_1 + r_2 = n, \end{cases}$$

where J is the matrix $J = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$. Thus, if (5) holds, then

$$\operatorname{syz}_1(Q_{N_2}) = \operatorname{syz}_1(Q_{N_1})[N_1 - N_2].$$

Finally, we assume (1) – (5) hold. These conditions guarantee that there exist integers w_1, w_2, w_3 with

$$\operatorname{soc} Q_{N_2} \cong \operatorname{soc} Q_{N_1}(-w_1), \quad \operatorname{syz}_1 Q_{N_2} \cong \operatorname{syz}_1 Q_{N_1}(-w_2), \quad \text{and} \quad \mathbb{F}_{2,\geq 1} \cong \mathbb{F}_{1,\geq 1}(-w_3).$$

The complex $\mathbb{F}_{i,\geq 1}$ is a resolution of $\operatorname{syz}_1 Q_{N_i}$; so $w_2 = w_3$, and the isomorphism (9.7) shows that $w_1 = w_3$. We have already seen that when condition (5) is in effect, then $w_2 = N_2 - N_1$. $\square$

Example 9.8. Fix a positive integer e and a field $\boldsymbol{k}$ of characteristic $p > 0$. Take $n = p^e + 1$ and N to be any integer which is relatively prime to n. Let R be the ring $\bar{R}_{\boldsymbol{k},n}$ and for each integer M, let Q_M be the R-module $\bar{Q}_{k,n,M}$ as described in (9.1). Let $t_1 \leq t_2$ be positive integers. We observe that

(9.9) $\operatorname{syz}_1 Q_{p^{t_2}N}$ is isomorphic to a shift of $\operatorname{syz}_1 Q_{p^{t_1}N}$ if and only if the integer $t_2 - t_1$ is divisible by e.

In other words, $\operatorname{syz}_1 Q_{p^iN}$ represent different isomorphism classes, even after shifting, for $0 \leq i \leq e-1$; but $\operatorname{syz}_1 Q_{p^{k+e}N}$ is isomorphic to a shift of $\operatorname{syz}_1 Q_{p^kN}$ for all integers $k \geq 0$.

Assertion (9.9) follows from Observation 9.4, where it is shown that $\operatorname{syz}_1 Q_{p^{t_2}N}$ is isomorphic to a shift of $\operatorname{syz}_1 Q_{p^{t_1}N}$ if and only if $p^{t_2}N \equiv \pm p^{t_1}N \mod n$. We have arranged the data so that $p^e \equiv -1 \mod n$; but if $0 \leq t_1 < t_2 \leq e-1$, then p^{t_2} is not congruent to $\pm p^{t_1} \mod n$.

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