TERENCE TAO'S "AN EPSILON OF ROOM" CHAPTER 3 EXERCISES

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1. Exercise 1.3.1

We merely consider the inclusion $f \mapsto f$, viewed as an element of $L^p(X, \chi, \mu)$, where all nonmeasurable subnull sets are given measure 0.

Linearity is trivial. Surjectivity is also immediate, since the preimage is merely the function f itself. It remains to prove injectivity, so assume $f \mapsto 0$. Then, $\text{Supp}(f) \subset A$ for some null set A. Integrating over A, we find that $f = 0$ a.e, which proves injectivity.

2. Exercise 1.3.2

(i). Note that whenever $p < 1$ and $|x| \leqslant 1$, $|x| \leqslant |x|^p$. Then,

$$\begin{aligned}
1 &= \frac{|f|}{|f|+|g|} + \frac{|g|}{|f|+|g|} \\
&\leqslant \Big(\frac{|f|}{|f|+|g|}\Big)^p + \Big(\frac{|g|}{|f|+|g|}\Big)^p \\
&= \frac{|f|^p + |g|^p}{(|f|+|g|)^p} \\
&\implies (|f|+|g|)^p \leqslant |f|^p + |g|^p
\end{aligned}$$

Combining this with the triangle inequality and integrating,

$$||f+g||_p^p \leqslant ||f||_p^p + ||g||_p^p$$

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(ii). Note first that $|x| \mapsto |x|^p$ is concave for $p < 1$. By homogeneity, it is of no loss of generality to assume that $||f||_p + ||g||_p = 1$, so that for $||F||_p = ||G||_p = 1$,

$$f = (1-\theta)F, \quad g = \theta G, \quad \theta \in (0,1)$$

So that by concavity,

$$|(1-\theta)F + \theta G|^p \geqslant (1-\theta)|F|^p + \theta|G|^p$$

Integrating yields

$$||f+g||_p \geqslant 1 = ||f||_p + ||g||_p$$

As asserted.

(iii). We may again assume $||f||_p + ||g||_p = 1$, so that for $||F||_p = ||G||_p = 1$,

$$f = (1-\theta)F, \quad g = \theta G, \quad \theta \in (0,1)$$

By part (i),

$$||f+g||_p \leqslant \big((1-\theta)^p + \theta^p\big)^{1/p}$$

Since $|x| \mapsto |x|^{1/p}$ is convex,

$$\begin{aligned}\big((1-\theta)^p + \theta^p\big)^{1/p} &\leqslant (1-\theta)\cdot(1-\theta)^{1-1/p} + \theta \cdot \theta^{1-1/p} \\ &= (1-\theta)^{2-1/p} + \theta^{2-1/p}\end{aligned}$$

Optimizing in θ, we find that the minimum is achieved for $\theta = 1/2$, implying

$$\begin{aligned}||f+g||_p &\leqslant \left(\frac{1}{2}\right)^{2-1/p} + \left(\frac{1}{2}\right)^{2-1/p} \\ &= 2^{1/p-1}\end{aligned}$$

Which yields our optimal constant as $2^{1/p-1}$, as asserted.

(iv). Note that by strict convexity/concavity, equality holds if and only if $g = cf$, $c \in \mathbb{R}$. In the $p = 1$ case, we merely require that f and g always have the same sign (that is, $fg \geqslant 0$).

3. Exercise 1.3.3

Suppose first that $|| \cdot ||$ is a norm. If B denotes our unit ball, let $x,\ y \in B$, $t \in (0, 1)$:

$$\begin{aligned} ||(1-t)x + ty|| &\leqslant (1-t)||x|| + t||y|| \\ &\leqslant 1 - t + t = 1 \end{aligned}$$

So that B is convex. Conversely, we merely use contraposition. Then there exist points x and $y \in B$ such that the triangle inequality does not hold. By homogeneity, we may assume that $x = (1-t)x'$, $y = ty'$ for $x',\ y' \in \partial B$ and that $||x|| + ||y|| = 1$. If we consider the line segment through x' and y', we see that for $\theta = t$,

$$||(1-t)x' + ty'|| = ||x + y|| > ||x|| + ||y|| = 1$$

So that B is not convex, whence the result.

4. Exercise 1.3.4

Define $A_0 := [1, \infty]$, $A_n := \left[\frac{1}{n+1}, \frac{1}{n}\right]$. Noting that $\text{Supp}(f) = \text{Supp}(|f|)$, we see:

$$\text{Supp}(f) = \bigcup_{n=0}^{\infty} |f|^{-1}(A_n)$$

Yielding σ-finiteness.

5. Exercise 1.3.5

If $||f||_\infty = 0$, $f \equiv 0$ trivially. Assume now that $f \not\equiv 0$. For sufficiently small $\epsilon > 0$, consider

$$S_\epsilon := \{x \mid |f(x)| \geqslant ||f||_\infty - \epsilon\}$$

By the previous problem, we may assume without loss of generality that $\mu(S_\epsilon) < \infty$. Then,

$$\begin{aligned} ||f||_p &\geqslant \Big(\int_{S_\epsilon} (||f||_\infty - \epsilon)^p d\mu \Big)^{1/p} \\ &= (||f||_\infty - \epsilon)\mu(S_\epsilon)^{1/p} \end{aligned}$$

Taking the limit inferior,

$$\liminf_{p\to\infty} ||f||_p \geqslant ||f||_\infty - \epsilon$$

As $\epsilon > 0$, is arbitrary, $\liminf_{p\to\infty} \geqslant ||f||_\infty$. Now, as $f \in L^{p_0} \cap L^\infty$, Hölder's inequality yields for all $p > p_0$:

$$||f||_p \leqslant ||f||_\infty^{\frac{p-p_0}{p}} ||f||_{p_0}^{\frac{p_0}{p}}$$

Letting $p \to \infty$,

$$\limsup_{p\to\infty} ||f||_p \leqslant ||f||_\infty$$

Combining with the reverse inequality, we deduce

$$\lim_{p\to\infty} ||f||_p = ||f||_\infty$$

As asserted. Suppose now that $f \notin L^\infty$. Then, for every $n > 0$, there exists a set E_n and $\epsilon > 0$ such that $\mu(E_n) \geqslant \epsilon > 0$ and $f(x) > n$ for every $x \in T_\epsilon$. We see:

$$\begin{aligned} n\epsilon^{1/p} &= n\mu(E_n)^{1/p} \\ &< \Big(\int_{E_n} |f|^p d\mu \Big)^{1/p} \leqslant ||f||_p \end{aligned}$$

Letting $p \to \infty$,

$$n < \liminf_{p\to\infty} ||f||_p$$

for all integers n. Hence, letting $n \to \infty$, $\lim_{p\to\infty} ||f||_p = \infty$.

6. Exercise 1.3.6

Define $d(f, g) := ||f - g||$.

Homogeneity:

$$\begin{aligned} d(cf, cg) &= ||cf - cg|| \\ &= |c|||f - g|| = |c|d(f, g) \end{aligned}$$

Triangle Inequality:

$$\begin{aligned} d(f, h) &= ||f - h|| \\ &\leqslant ||f - g|| + ||g - h|| = d(f, g) + d(g, h) \end{aligned}$$

Separation:

$$d(f, g) = 0 \iff ||f - g|| = 0 \iff f = g$$

Symmetry:

$$\begin{aligned} d(f, g) &= ||f - g|| \\ &= ||g - f|| = d(g, f) \end{aligned}$$

Translation Invariance:

$$\begin{aligned} d(f + h, g + h) &= ||(f + h) - (g + h)|| \\ &= ||f - g|| = d(f, g) \end{aligned}$$

Conversely, suppose we have a translation invariant homogeneous metric $d : V \times V \to [0, \infty]$. Define $||f|| := d(f, 0)$. This choice is clearly unique, since any definition with respect to a nonzero basepoint loses homogeneity. It remains only to show the triangle inequality:

$$\begin{aligned} ||f + g|| &= d(f + g, 0) \\ &= d(f, -g) \\ &\leqslant d(f, 0) + d(0, -g) \\ &= d(f, 0) + d(g, 0) = ||f|| + ||g|| \end{aligned}$$

So that $|| \cdot ||$ defines a unique norm.

7. Exercise 1.3.7

Assume first that V is complete. Given an absolutely convergent series $\sum_{n=1}^{\infty} ||f_n||$, the sequence of partial sums s_N is Cauchy. But

$$||\sum_{n=1}^{\infty} f_n|| \leqslant \sum_{n=1}^{\infty} ||f_n||$$

So that $\sum_{n=1}^{N} f_n$ is bounded by s_N. As the s_N are Cauchy, we deduce that $\sum_{n=1}^{N} f_n$ is Cauchy. By completeness, this sequence converges, so that $\sum_{n=1}^{\infty} f_n$ exists.

Conversely, suppose that any absolutely convergent sum converges conditionally. Let f_n be a Cauchy sequence, and extract a subsequence f_{n_k} such that

$$||f_{n_{k+1}} - f_{n_k}|| < \frac{1}{2^k}$$

for $k \in \mathbb{N}$. Then, obviously $\sum_{k=1}^{\infty} ||f_{n_{k+1}} - f_{n_k}||$ converges. By assumption, this implies $\sum_{k=1}^{\infty} f_{n_{k+1}} - f_{n_k}$ converges as well. But this sum in telescoping with limit $\lim_{k\to\infty} f_{n_k} - f_{n_1}$, so we deduce that $f_{n_k} \to f$ for some f. It remains to show that $f_n \to f$, but as f_n is Cauchy:

$$||f_n - f|| \leqslant ||f_n - f_{n_k}|| + ||f_{n_k} - f|| \to 0$$

as $n,\ k \to \infty$, so that $f_n \to f$, implying that every Cauchy sequence is convergent, so V is complete.

8. Exercise 1.3.8

If $f \in L^{\infty}$, f is essentially bounded. Hence there exists a sequence of simple functions increasing to f (by the simple approximation theorem), and density is trivial. Note that the space of simple functions with finite measure is not dense in L^{∞}. To see this, merely consider the characteristic function $\chi_{\mathbb{R}}$. This has $||\chi_{\mathbb{R}}||_{\infty} = 1$. Choosing a sequence of simple functions s_n with finite support, we see that for every n, s_n vanishes outside of some sufficiently large set, so that $||\chi_{\mathbb{R}} - s_n||_\infty = 1$ for all n. Therefore, this set cannot possibly be dense.

9. Exercise 1.3.9

Enumerate the generators of our σ algebra Ω by $B = \{E_1, E_2, \ldots\}$. One immediately sees that

$$\{\sum_{\text{finite}} \chi_{E_n} \mid E_n \in B\}$$

is dense in the space of characteristic function. Since $\mathbb{Q}$ is also dense in the reals, we see that

$$S := \operatorname{span}_{\mathbb{Q}}\{\chi_{E_n} \mid E_n \in B\}$$

is dense in the space of simple functions with finite measure support, denoted S_0, which in turn is dense in L^p. But this implies

$$\overline{S} = S_0 \quad \text{and} \quad \overline{S}_0 = L^p$$

From which we immediately see that S is dense in L^p as well. But S is countable, so we deduce that L^p is separable.

For $p = \infty$, consider the family

$$C := \{\chi_{[-r,r]} \mid r \in \mathbb{R}^+\}$$

Then for any two distinct $x,\ y \in C$, $||x - y||_\infty = 1$. But then L^∞ is certainly not separable, as no sequence of distinct elements could ever converge to an element of C.

10. Exercise 1.3.10

Young's inequality is established by concavity, and the case of equality for strict concavity will occur precisely when $a^p = cb^q$. By homogeneity, we may assume $||f||_p = ||g||_q = c = 1$. Then,

$$\begin{aligned} |f(x)g(x)| &= \frac{1}{p}|f(x)|^p + \frac{1}{q}|g(x)|^q \\ &\implies ||fg||_1 = 1 = ||f||_p \cdot ||g||_q \end{aligned}$$

Yielding the result.

11. Exercise 1.3.11

Let $f \in L^p$, $q < p$. Then

$$\begin{aligned} ||f||_q &\leqslant \Big(\int_E 1 d\mu \Big)^{1/q-1/p} \Big(\int_E |f|^p d\mu \Big)^{1/p} \\ &= \mu(E)^{1/q-1/p} ||f||_p \end{aligned}$$

So $f \in L^q$. By the previous problem, equality occurs when f is a constant.

12. Exercise 1.3.12

Since $q > p$, we may apply the reverse H ólder inequality to find

$$||f||_p \geqslant ||f||_q \mu(E)^{1/p-1/q}$$

By assumption, $\mu(E) \geqslant m$ for every E in our σ-algebra, so we rearrange the above inequality to find

$$||f||_q \leqslant m^{1/q-1/p} ||f||_p$$

So that $f \in L^q$ if $f \in L^p$. Equality holds again for $f \equiv \text{const}$.

13. Exercise 1.3.13

We have:

$$\begin{aligned}\int_X |f(x)|^p d\mu &= \int_X |f|^{p\theta}|f|^{(1-\theta)p} d]mu \\ &\leqslant \Big(\int_X \big(|f|^{p\theta}\big)^{\frac{p_1}{p\theta}} d\mu\Big)^{\frac{p\theta}{p_1}} \cdot \Big(\int_X \big(|f|^{(1-\theta)p}\big)^{\frac{p_0}{(1-\theta)p}} d\mu\Big)^{\frac{p(1-\theta)}{p_0}} \\ &= ||f||_{p_1}^{p\theta}||f||_{p_0}^{p(1-\theta)}\end{aligned}$$

Taking pth roots in the above,

$$||f||_p \leqslant ||f||_{p_1}^{\theta}||f||_{p_0}^{1-\theta}$$

whence the result.

14. Exercise 1.3.14

By Hölder's,

$$||f||_p^p \leqslant \mu(E)^{1-p/p_0}||f||_{p_0}^p$$

so that

$$\limsup_{p\to 0} ||f||_p^p \leqslant \mu(E)$$

For the reverse inequality,

$$\begin{aligned}\liminf_{p\to 0}\int_X |f|^p d\mu &\geqslant \liminf_{p\to 0}\int_E |f|^p d\mu \\ &\geqslant \int_E \liminf_{p\to 0} |f|^p d\mu \quad \text{(Fatou's)} \\ &= \int_E d\mu = \mu(E)\end{aligned}$$

Hence we conclude that $\lim_{p\to 0} ||f||_p^p = \mu(E)$.