TERENCE TAO'S "AN EPSILON OF ROOM" CHAPTER 2 EXERCISES

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1. Exercise 1.2.1

Let s_n be an increasing sequence of simple functions tending to g. Then,

$$\begin{aligned}
\int_X s_n dm_f &= \int_X \sum_i a_i \chi_{X_i} dm_f \\
&= \sum_i a_i \int_{X_i} f dm \\
&= \sum_i a_i \int_X \chi_{X_i} f dm \\
&= \int_X s_n f dm
\end{aligned}$$

Letting $n \to \infty$, the monotone convergence theorem yields

$$\int_X g dm_f = \int_X g f dm$$

2. Exercise 1.2.2

If $f = g$ a.e, then,

$$\int_X (f - g) dm = 0 \implies \int_X f dm = \int_X g dm$$

So that $m_f = m_g$. Conversely, suppose $m_f = m_g$. By σ-finiteness, it is of no loss of generality to assume $m(X) < \infty$. If $f \neq g$ a.e, we can find a set of positive measure such that $f - g > \epsilon > 0$, in which case we

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would see that $m_f - m_g$ would also have to be positive, a contradiction. The result follows.

Choose any singleton set X with the discrete topology, and define $\mu(X) = \infty$ and $\mu(\varnothing) = 0$. Given any two distint $f,\ g : X \to Y$, we see that $m_f = m_g = \infty$, but f and g are not equal.

3. Exercise 1.2.3

Define $\frac{d\mu}{dm}(x) := f(x)$. Let $\epsilon > 0$, and choose h small enough such that $|f(x+h) - f(x)| < \epsilon$. Consider then:

$$\begin{aligned}
|\mu([x, x+h] - f(x)h| &= |\int_{[x,x+h]} d\mu - f(x)m([x, x+h])| \\
&= |\int_{[x,x+h]} f(y)dm(y) - \int_{[x,x+h]} f(x)dm(y)| \\
&\leqslant \int_{[x,x+h]} |f(y) - f(x)|dm(y) \\
&< \epsilon h \to 0
\end{aligned}$$

As $\epsilon \to 0$. Hence, by definition of derivative, $f(x) = \frac{d}{dx}\mu([0, x])$.

4. Exercise 1.2.4

Let X be at most countable with measure μ on the discrete σ-algebra. Using the fact that $\int_{\{x\}} d\# = 1$, choose $A \subset X$ arbitrary:

$$\begin{aligned}
\int_A d\mu &= \int_A \int_{\{x\}} d\# d\mu \\
&= \int_{\{x\}} \int_A d\mu d\# \quad \text{(Fubini's)} \\
&= \int_{\{x\}} \sum_{x \in A} \int_{\{x\}} d\mu d\# \\
&= \sum_{x \in A} \int_{\{x\}} \int_{\{x\}} d\mu d\#
\end{aligned}$$

From which we immediately deduce that $\frac{d\mu}{d\#}(x) = \mu(\{x\})$, so our derivative exists.

5. Exercise 1.2.5

Let μ be a signed measure. Decompose $X = X_+ \cup X_-$ as asserted by the Hahn decomposition theorem. Define $\mu_+ := \mu|_{X_+}$, $\mu_- := -\mu|_{X_-}$. Clearly $\mu = \mu_+ - \mu_-$, it remains to prove uniqueness. Suppose that two such decompositions exists, that is,

$$\mu_+ - \mu_- = \eta_+ - \eta_-$$

for some other measures $\eta_+,\ \eta_-$. By evaluating on all subsets on X_+ and X_-, mutual singularity guarantees that $\mu_+ = \eta_+$ and $\mu_- = \eta_-$, so uniqueness is immediate.

6. Exercise 1.2.6

Suppose for sake of contradiction there exists some other measure η such that

$$-|\mu| < -\eta \leqslant \mu \leqslant \eta < |\mu|$$

Evaluating on X_+ and X_-, respectively, we find that

$$\mu(X_+) \leqslant \eta(X_+) < \mu_+(X_+) = \mu(X_+)$$

$$-\mu(X_-) = -\mu_-(X_-) < -\eta(X_-) \leqslant \mu(X_-)$$

These contradictions imply that no such measure can exist.

7. Exercise 1.2.7

Since μ is bounded by $|\mu|$, μ being infinite implies that $|\mu|$ is infinite. Conversely, if μ is infinite, then either μ_- or μ_+ is infinite, in which case μ must also be infinite.

Taking contrapositives yields $|\mu| < \infty \iff \mu_-,\ \mu_+ < \infty$.

8. Exercise 1.2.8

Suppose that μ is σ-finite. Choose a sequence of finite measure subsets E_n increasing to X, so that there exists f_n such that $\int_{\bigcup_{i=1}^n} d\mu = \int_{\bigcup_{i=1}^n} f_n dm + \mu_{ns}$. Letting $n \to \infty$, suppose $_n \to f$. The monotone convergence theorem yields

$$\int_X d\mu = \int_X f dm + \mu_s$$

We see that $\mu_s \perp m$, since $\mu_{ns} \perp m$ for every n, whence the result follows.

9. Exercise 1.2.9

Define $D := \{x \in X \mid \mu(\{x\}) > 0\}$. By σ-finiteness, this set is countable, so we can enumerate $D = \{x_1, x_2, \dots\}$. If D is empty, we can take $\mu_{pp} \equiv 0$ and employ the Radon-Nikodym-Lebesgue theorem.

Assume now that $D \neq \varnothing$, and define

$$\mu_{pp} := \sum_n \mu(\{x_n\})\chi_{\{x_n\}}$$

and set $\mu_c = \mu - \mu_{pp}$. By construction,

$$\begin{aligned}\mu_c(\{x\}) &= \mu(\{x\}) - \mu_{pp}(\{x\}) \\ &= \mu(\{x\}) - \mu(\{x\}) = 0\end{aligned}$$

So that μ_c is continuous. Now, by the Radon-Nikodym-Lebesgue theorem, there exists some $f \in L^1(X, m)$ such that $\mu_c = m_f + \mu_{sc}$. Note that m_f is a continuous measure, and we deduce that μ_{sc} must be continuous with $m_f \perp \mu_{sc}$. Hólder's inequality immediately gives that $\mu_f \ll m$, so we can set $m_f := \mu_{ac}$, and deduce that

$$\mu = \mu_{ac} + \mu_{sc} + \mu_{pp}$$

As desired.

10. Exercise 1.2.10

Assume first that the function $f(x) := \mu([0,x])$ is continuous. In view of the Hahn decomposition theorem, we may assume that μ is unsigned. Since $\{x\} \subset [x, x+h]$ for every $h > 0$,

$$\mu(\{x\}) \leqslant \mu([x, x+h])$$

for all h. Letting $h \to 0$, we find that $\mu(\{x\}) = 0$.

Conversely, argue by contraposition. If $\mu([0, x]0$ is not continuous at some $x \in X$, we can find $\epsilon > 0$ such that

$$\mu([x, x+h]) \geqslant \epsilon$$

for all $h > 0$. Discretizing,

$$\lim_{n\to\infty} \mu([x, x+1/n]) = \mu(\{x\}) \geqslant \epsilon > 0$$

so that $\mu(\{x\}) > 0$ for some $x \in X$.