# REAL ANALYSIS HW 8

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## 1. Problem 1

For convenience, we can rewrite

$$\omega(x) = \lim_{r \to 0^+} \sup_{x', x'' \in B(x,r)} |f(x') - f(x'')|$$

Then, suppose $\omega(x) < c$. We can find $\epsilon > 0$ with $\omega(x) + \epsilon < c$, and by definition of limit, there exists $r > 0$ such that $\sup_{z,y \in B(x,r)} |f(y) - f(z)| - \omega(x) < \epsilon$ (noting that the quantity on the left will always be positive since it is decreasing with respect to $r$).

Fix this $r$, and let $y \in B(x,r)$. There exists $r'$ such that $B(y, r') \subset B(x, r)$. Using this:

$$\begin{aligned} \omega(y) &\leqslant \sup_{x', x'' \in B(y, r')} |f(x') - f(x'')| \\ &\leqslant \sup_{x', x'' \in B(x,r)} |f(x') - f(x'')| < \omega(x) + \epsilon < c \end{aligned} \tag{1.1}$$

Thus, we've found that there exists $r > 0$ such that for all $y \in B(x,r)$, $\omega(y) < c$. Then by definition, $\{x : \ \omega(x) < c\}$ is open.

## 2. Problem 2

Since $m(F) = 0$, there exists an open cover $\{I_n\}$ with $\sum_n |I_n| < \epsilon$ for all $\epsilon > 0$. Since $F$ is closed and bounded, it is compact and we can extract a finite subcover $\{I_1, \dots, I_k\}$. Enumerating the end points,

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we can construct a partition from the $I_j$ and their complements. Since $f := \mathbf{1}_F$ vanishes outside of this cover, we only need to consider taking the upper sum over all $I_k$, since the lower sum will always be 0.

Then, denoting our partition $P$:

$$U(f,P) = \sum_{j=1}^{k} 1 \cdot |I_j| < \epsilon$$

By construction of our $I_j$. Note, we've used that the max of the characteristic function is just 1. Then, we've shown that for all $\epsilon > 0$, there exists a partition $P$ such that $U(f,P) - L(f,P) < \epsilon$. By definition, $f$ is integrable.

## 3. Problem 3

Since $E$ is dense in $[a,b]$ (ie, $\overline{E} = [a,b]$), every neighborhood of any $x \in [a,b]$ must intersect $E$. From this we can immediately deduce that $L(f,P) = 0$ for any given partition $P$, since $f = 0$ on $E$. Since $P$ was arbitrary, in general $L(f) = 0$ (denoting the lower integral).

By the criterion for integrability, $U(f) = L(f) = \int_a^b f$. Since $L(f) = 0$ and $f$ was given to be integrable, we conclude that $\int_a^b f = 0$, as asserted.

## 4. Problem 4

Let $\epsilon > 0$ with $f_n \to f$. By uniform continuity of our sequence and continuity of each $f_n$, we can find $\delta$ and $N$ such that for all $n > N$ and all $|x - y| < \delta$,

$$|f(x) - f_n(x)| < \epsilon/3$$

$$|f(y) - f_n(y)| < \epsilon/3$$

$$|f_n(x) - f_n(y)| < \epsilon/3$$

Then, for all $n > N$ and all $|x - y| < \delta$,

$$\begin{aligned} |f(x) - f(y)| &\leqslant |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(x) - f_n(y)| \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon \end{aligned} \tag{4.1}$$

Where the above uses the triangle inequality. Then, $f$ is continuous by definition.

## 5. Problem 5

First, letting $E := [a, b]$, we can write $f(x) = \int_E g\mathbf{1}_{[a,x]}$. Without loss of generality, suppose that $g$ is nonnegative (else just split into positive/negative parts). Let $\epsilon > 0$ and suppose $x_n \to x$. Note that $|f(x)| \leqslant \int_E g$, and hence by the dominated convergence theorem:

$$\lim_{n\to\infty} \int_E g\mathbf{1}_{[a,x_n]} = \int_E \lim_{n\to\infty} g\mathbf{1}_{[a,x_n]} = \int_E g\mathbf{1}_{[a,x]}$$

Then we see $f(x_n) \to f(x)$ for all sequences $x_n \to x$, so that $f$ is continuous.