REAL ANALYSIS HOMEWORK 2

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1. Problem 1

Enumerate the elements of A as $\{a_1, a_2, \dots\}$. This is possible since A is countably infinite. Now, define $f : \mathbb{N} \to A \cup \{a\}$ by $f(1) = a$ and $f(n) = a_{n-1}$ for $n \geq 2$. Then, f is a bijection since it has left and right inverse defined by $f^{-1}(a) = 1$ and $f^{-1}(a_n) = n + 1$, so that $A \cup \{a\}$ is countable by definition.

2. Problem 2

Since A and B are equipotent, there exists a bijection $f : A \to B$. Define $g : A \times C \to B \times C$ such that $(a, c) \mapsto (f(a), c)$. g is bijective since we have left and right inverse g^{-1} sending $(b, c) \mapsto (f^{-1}(b), c)$ (note f^{-1} exists and is well defined since f is bijective). Then, by definition, $A \times C$ and $B \times C$ are equipotent.

3. Problem 3

Enumerate the rational points in $(0, 1)$ as $\{r_0,\ r_1,\ r_2,\ \dots\}$ (this is possible since $\mathbb{Q}$ is countable). Define $f : (0, 1) \to [0, 1]$ by first sending $r_0 \mapsto 0$ and $r_1 \mapsto 1$. Then, define $f(r_n) = r_{n-2}$ for $n \geq 2$ and $f\big|_{(0,1)\setminus\mathbb{Q}} \equiv \text{id}$. This is bijective since we have right and left inverse f^{-1} defined by $f(0) = r_0$, $f^{-1}(1) = r_1$, $f^{-1}(r_n) = r_{n+2}$ and $f(q) = q$ for irrational q.

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4. Problem 4

Suppose there exists a bijection $f : \mathbb{N} \to 2^{\mathbb{N}}$. Then, consider the set $A := \{k \in \mathbb{N} : k \notin f(k)\}$. This is clearly a subset of the integers, so that $A \in 2^{\mathbb{N}}$.

Since f is a bijection there corresponds an integer n such that $f(n) = A$. We now have two cases: either $n \in f(n)$ or $n \notin f(n)$.

Suppose first that $n \in f(n)$. Then, by definition, $n \notin f(n)$, so this case is impossible.

Suppose now that $n \notin f(n)$. Then, again by definition of A, $n \in A \implies n \in f(n)$, which is again a contradiction. Thus, A has no preimage so that f cannot possibly be surjective, contradicting bijectivity. This tells us that $2^{\mathbb{N}}$ has cardinality strictly "larger" than $\mathbb{N}$, and is hence uncountable.

5. Problem 5

Let O_i be a finite collection of open sets and consider $\bigcap_i O_i$. If $\bigcap_i O_i = \varnothing$, then their intersection is trivially open, so assume $\bigcap_i O_i \neq \varnothing$. Choose $x \in \bigcap_i O_i$ and let δ_i be the finite set of positive real numbers such that $B_{\delta_i}(x) \subset O_i$ [1]

Then, set $\delta = \min_i\{\delta_i\}$. This δ exists and is > 0 since the set of δ_i is a finite set with each $\delta_i > 0$. Then, by construction, $B_\delta(x) \subset B_{\delta_i}(x) \subset O_i$ for all i, so that $B_\delta(x) \subset \bigcap_i O_i$. By definition, we have that $\bigcap_i O_i$ is open as well.

Now, let $\{O_i\}_{i\in I}$ be an arbitrary collection of open sets and let $x \in \bigcup_{i\in I} O_i$ (we are again assuming this is nonempty, since the other case is trivial). Then, for some $j \in I$, $x \in O_j$, and by openness there

[1] $B_\delta(x)$ will denote the open ball of radius δ centered at x.

corresponds δ_j such that $B_{\delta_j}(x) \subset O_j \subset \bigcup_i O_i$. Then, again by definition, $\bigcup_i O_i$ is open.