REAL ANALYSIS HW 9

KELLER VANDEBOGERT

1. Problem 1

By the Vitali Covering Lemma, we can find a disjoint, finite family of sets $\{I_{k,1}\}_{k=1}^{n_1}$ with $m^*(E \backslash A_1) < 1$, where $A_1 := \bigcup_{k=1}^{n_1} I_k$.

Now consider the set $E \backslash A_1$. Then, $\mathcal{F}$ is still a Vitali cover of this. Hence we can find a disjoint collection $\{I_{k,2}\}_{k=1}^{n_2}$ with $m^*((E \backslash A_1) \backslash A_2) < 1/2$. Continuing inductively, at the nth step, $\mathcal{F}$ will remain a Vitali covering for $E \backslash (\bigcup_{i=1}^{n-1} A_i)$, so use the covering lemma to find another disjoint set $\{I_{k,n}\}_{i=1}^{m_n}$ such that $m^*((E \backslash (\bigcup_{i=1}^{n-1} A_i) \backslash A_n) < 1/n$.

Doing this for all n, we obtain a disjoint collection of sets $\{A_k\}_{k=1}^{\infty}$ such that $m^*(E \backslash \bigcup_{k=1}^{\infty} A_k) < 1/n$ for all n. Letting $n \to \infty$, we see that $m^*(E \backslash \bigcup_{k=1}^{\infty} A_k) = 0$, as desired.

2. Problem 2

By definition,

$$D^+(-f) = \limsup_{h \to 0^+} \frac{-f(x+h) + f(x)}{h} = -\liminf_{h \to 0^+} \frac{f(x+h) - f(x)}{h} = -D_+(f)$$

$$D^-(-f) = \limsup_{h \to 0^-} \frac{-f(x+h) + f(x)}{h} = -\liminf_{h \to 0^-} \frac{f(x+h) - f(x)}{h} = -D_-(f)$$

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3. Problem 3

Computing the quotients:

$$D^+f(0) = \limsup_{h\to 0^+} a\sin^2(1/h) + b\cos^2(1/h)$$
$$D_-f(0) = \liminf_{h\to 0^-} c\sin^2(1/h) + d\cos^2(1/h)$$

The maximum value of the first quotient is b, attained at the sequence of points $h_n = \frac{1}{2\pi n+\pi/2}$ for all n positive integers. Letting $n \to \infty$ we find that this sequence tends to 0^+ and yields the constant sequence $\{b\}$.

Similarly, the second quotient has a minimum at c attained at $h_n = \frac{-1}{2\pi n}$ for all positive integers n. Letting $n \to \infty$ again, we see $h_n \to 0^-$ and the difference quotient merey becomes the constant sequence $\{c\}$. Putting all of the above together:

$$D^+f(0) = b$$
$$D_-f(0) = c$$

4. Problem 4

(a). Differentiating yields:

$$f'(x) = 2x\sin(1/x^2) - \frac{2\cos(1/x^2)}{x}$$

(b). Concentrating on the second term of f' (since the first term is bounded), we can consider for all positive integers n, the set:

$$x \in \left[\sqrt{\frac{3}{6\pi n+\pi}}, \sqrt{\frac{3}{6\pi n-\pi}}\right] := E_n$$

It is obvious that on these intervals, $\cos(1/x^2) \geqslant 1/2$, so that on each E_n, $\cos(1/x^2)/x \geqslant 1/2\sqrt{2\pi n - \pi/3}$. Hence we have that

$$\sum_{n=1}^{N} 1/2\sqrt{2\pi n - \pi/3} \cdot \mathbf{1}_{E_n} \leqslant \cos(1/x^2)/x$$

Integrating our characteristic function,

$$\sum_{n=1}^{N} 1/2\sqrt{2\pi n - \pi/3} \cdot m(E_n) \to \infty$$

As $N \to \infty$. Then we see that this sequence of simple functions is majorized by $\cos(1/x^2)/x$ and is not integrable. Thus, $\cos(1/x^2)/x$ cannot be integrable and hence neither can f'.

5. Problem 5

(a). For all partitions P:

$$\sum_{i=0}^{n} |f(x_{i+1} - f(x_i)| \leqslant \sum_{i=0}^{n} L|x_{i+1} - x_i| = L(b-a)$$

Since this holds for all partitions P, taking the supremum yields that our total variation is bounded by $L(b-a)$ (and is hence of bounded variation).

(b). For all partitions P:

$$\sum_{i=0}^{n} |g(x_{i+1} - g(x_i)| = \sum_{i=0}^{n} |\int_{[x_i, x_{i+1}]} f| \leqslant \sum_{i=0}^{n} \int_{[x_i, x_{i+1}]} |f| = \int_{[a,b]} |f| < \infty$$

Where the final term is bounded since we were given that $f \in L^1(a,b)$. Since this is a bound for all partitions, we conclude that the total variation is also bounded, as asserted.