REAL ANALYSIS HOMEWORK 6

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1. Problem 1

(a). We have already shown that for measurable f, $f^{-1}\big([-\infty, c]\big)$ and $f^{-1}\big([c, \infty]\big)$ are measurable. Then, merely note that $\{c\} = [-\infty, c] \cap [c, \infty]$ to find:

$$f^{-1}(c) = f^{-1}\big([-\infty, c] \cap [c, \infty]\big) = f^{-1}\big([-\infty, c]\big) \cap f^{-1}\big([c, \infty]\big)$$

Then, noting that $\{x \in E : \ f(x) = c\} = f^{-1}(c)$, we see that this is the intersection of two measurable sets and is hence measurable.

(b). Consider $g : [0, 1] \to [-1, 1]$ with $g(x) = x$ for $x \in Y$, $g(x) = -x$ for $x \notin Y$, where Y is some nonmeasurable subset of $[0, 1]$ (we proved in class that such a set always exists). Explicitly, we could also say $g(x) = x \cdot (\mathbf{1}_Y - \mathbf{1}_{[-1,1]\setminus Y})$.

Then note that this is clearly a bijection and hence the preimage of every singleton is again a singleton in $[0, 1]$. But the measure of a singleton set is 0, so that $\{x \in [0, 1] : \ g(x) = c\}$ is always measurable. However, $\{x \in [0, 1] : \ g(x) \geqslant 0\}$ is precisely Y, which is nonmeasurable. Thus, g is not a measurable function and we are done.

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2. Problem 2

Suppose f is measurable, and following the hint define

$$\mathscr{A} := \{E \subset \mathbb{R} : \ f^{-1}(E) \text{ measurable}\}$$

Since f is measurable, it is clear that $\mathbb{R} \in \mathscr{A}$, and similarly for $\varnothing$.

Suppose that $A \in \mathscr{A}$. Then, consider the complement $\mathbb{R}\backslash A$: we see that $f^{-1}(\mathbb{R}\backslash A) = f^{-1}(\mathbb{R})\backslash f^{-1}(A)$. Since $f^{-1}(A)$ is measurable by assumption and the collection of measurable sets is a σ algebra, we see that $\big(f^{-1}(A)\big)^c$ is measurable and hence $A^c \in \mathscr{A}$.

Finally, given a countable collection $A_i \in \mathscr{A}$, $f^{-1}(\bigcup_i A_i) = \bigcup_i f^{-1}(A_i)$ is measurable as the countable union of measurable sets, again since the collection of measurable sets forms a σ algebra. Hence $\bigcup_i A_i \in \mathscr{A}$, and we have shown that indeed $\mathscr{A}$ is also a σ algebra.

Using this, it suffices to show that $\mathscr{A}$ contains all open sets. But this is clear: by definition of measurability, $f^((\infty, b) \cap f^{-1}(a, \infty)) = f^{-1}((a, b)) \in \mathscr{A}$, and hence $\mathscr{A}$ is a σ algebra containing every open set. But by definition, the Borel σ algebra is the smallest σ algebra containing every open set, and hence $\mathscr{A}$ contains the Borel σ algebra.

This then implies that $B \in \mathscr{A}$ for every Borel set B. In other words, $f^{-1}(B)$ is measurable for every Borel set B, which was to be proved.

3. Problem 3

Suppose $g = f$ a.e on E where f is measurable on E. Define $A := \{x \in E : \ f \neq g\}$. Then, $m(A) = 0$ by hypothesis and hence A is a measurable set. This of course then means that A^c is measurable since $E\backslash A$ is the difference of two measurable sets. On A^c however, $f \equiv g$ and hence g is certainly measurable since f is. Similarly, $g|_A$

is measurable since $f^{-1}|_A(B) \subset A$, and since A has measure 0, the preimage does as well and is hence measurable. Then, since $g|_A$ and $g|_{A^c}$ are both measurable, we conclude that $g|_E = g$ is measurable as well.

4. Problem 4

If f is measurable, then we have proved that $|f|$ is also a measurable function. Note that $x \mapsto x^p$ is continuous for $x \geqslant 0$, $p > 0$. Since the composition $\phi \circ g$ is continuous for g measurable and ϕ continuous, we see that $|f| \mapsto |f|^p$ is the continuous image of a measurable function, hence measurable.