REAL ANALYSIS HOMEWORK 10

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1. Problem 1

(a). Note that $f = g$ a.e on E is equivalent to saying $\int_E |f - g| = 0$. Hence, if $f = g$ a.e and $g = h$ a.e,

$$\int_E |f - h| \leqslant \int_E |f - g| + \int_E |g - h| = 0$$

So that $\int_E |f - h| = 0$ as well, showing that $f = h$ a.e.

(b). Suppose $f_1 = f_2$ and $g_1 = g_2$ a.e on E. Then:

$$\int_E |f_1 + g_1 - (f_2 + g_2)| \leqslant \int_E |f_1 - f_2| + \int_E |g_1 - g_2| = 0$$

So that $f_1 + g_1 = f_2 + g_2$ a.e.

(c). Suppose that $|f| \leqslant M$ and $|g| \leqslant N$ a.e on E. Then,

$$\int_E (M + N) - |f + g| \geqslant \int_E M - |f| + \int_E N - |f| \geqslant 0$$

And hence $|f + g| \leqslant M + N$ a.e.

2. Problem 2

The case $p_1 = 1$, $p_2 = \infty$ follows easily since

$$\int_E |f|dx \leqslant ||f||_\infty m(E) < \infty$$

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Hence, if $f \in L^\infty(E)$, we see that its L^1 norm must also be bounded. For the remaining case, find t such that $1/p_1 - 1/p_2 = 1/t$. Then, $1/p_2 + 1/t = 1/p_1$, and we can apply the general Hölder's inequality:

$$||f||_{p_1} \leqslant ||f||_{p_2}\big(m(E)\big)^{1/t}$$

Since $m(E) < \infty$, we see that $f \in L^{p_2}(E)$ implies $f \in L^{p_1}(E)$, which was to be proved.

3. Problem 3

(a). Let $\epsilon > 0$ and choose $A \subset E$ such that $m(A) < \epsilon/M$. By Hölder's inequality:

$$\int_A |f|dx \leqslant ||f||_p m(A) < \epsilon$$

So this is a uniformly integrable collection.

(b). Let $E = [0,1]$ and consider the family $\{f_n := n\mathbf{1}_{[0,1/n]}\}$. Certainly this collection is bounded in L^1 norm, as $\int_E f_n = 1$ for all n. However, given $1 > \epsilon > 0$ we can find N such that $1/N < \delta$ for any positive δ. Then we see that $[0,\delta]$ has measure δ, but $\int_{[0,\delta]} f_n = 1 > \epsilon$ for all $n \geqslant N$. Thus this collection cannot possibly be uniformly integrable.

4. Problem 4

Let $x_n \to x$. Then we can find N such that for all $n > N$, $||x_n - x|| < \epsilon/2$. Then, let m, $n > N$. By the triangle inequality:

$$||x_n - x_m|| \leqslant ||x_n - x|| + ||x - x_m|| < \epsilon$$

And hence x_n is Cauchy.