DIFFERENTIAL MANIFOLDS HOMEWORK 8

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1. Problem 1

We proceed to find a condition for which the orthogonal complement of a vector field is an involutive (and hence integrable) distribution.

First, associate to a vector $v \in \mathbb{R}^n$ its dual 1-form, which we will denote ω. Then the orthogonal complement, $v^\perp$, is precisely $\text{Ker}(\omega)$. However, once in this form, the problem becomes trivial. We have that (by Theorem 11.18 in Lee) that $\text{Ker}(\omega)$ is involutive if and only if the ideal $\langle \omega \rangle$ is a differential ideal.

Put more precisely, we have that $\mathrm{d}\omega \in \langle \omega \rangle$, and taking the wedge product with ω, we find:

$$\omega \wedge \mathrm{d}\omega = 0$$

Then it suffices to show that the above condition is in fact equivalent to $\langle \omega \rangle$ being a differential ideal. However, this follows immediately, since $\omega \wedge \mathrm{d}\omega = 0$ if and only if ω and $\mathrm{d}\omega$ are linearly dependent.

Hence we have derived a necessary and sufficient condition for the distribution spanned by the orthogonal complement of a vector v to be involutive. Namely, $\omega \wedge \mathrm{d}\omega = 0$, for the associated dual form ω of v.

For a case of classical interest, we can work in 3 dimensions and find that:

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$$\omega \wedge \mathrm{d}\omega = \langle v, \mathrm{curl}\ v \rangle \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z = 0$$

And hence $v^{\perp}$ is involutive if and only if $\langle v, \mathrm{curl}\ v \rangle = 0$.