DIFFERENTIAL MANIFOLDS HW 7

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1. Problem 1

(a). We have a natural action by considering a transformation taking the pair $(r+\mathbb{R}u, \mathbb{R}u) \mapsto (r - \langle r, u\rangle u + \mathbb{R}, \mathbb{R}u)$. Then, these two oriented lines are in fact identical since we have only shifted by a vector parallel to u.

Because of this, it is to be expected that the action by the Euclidean group commutes with the above identification between (r, u) and $(r - \langle r, u\rangle u, u)$, and since there is an obvious action sending $(r, u) \mapsto (Ar + c + \mathbb{R}Au, Au)$, composing our actions becomes (denoting elements of the Euclidean group as (A, c) where $A \in \mathrm{SO}(3)$, $c \in \mathbb{R}^3$):

$$(A, c) \cdot (r, u) = (Ar + c - \langle Ar + c, Au\rangle Au, Au)$$

And this is our natural action on Y.

(b). To show transitivity, let $x_0 := (0, e_3) \in \mathbb{R}^6$ with $e_3 = (0, 0, 1)$. Then, it is possible to show $G(x_0) = Y$. Let $(r, u) \in Y$ be arbitrary. Then, we can complete u to an orthonormal basis $\{u_1, u_2, u\}$. Set $A := (u_1, u_2, u)$. It is readily seen that this is a matrix in $\mathrm{SO}(3)$. Also, set $c := r$. Computing our action:

$$(A, c) \cdot (0, e_3) = (r, u)$$

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Hence this is transitive. Now let us show that this is Hamiltonian. We have the natural 1-form $\omega(\delta x) = -k\langle r, \delta u\rangle$ (k is any constant), where $x = (r, u)$. Taking the exterior derivative:

$$\mathrm{d}\omega(\delta x, \delta' x) = k(\langle \delta u, \delta' r\rangle - \langle \delta' u, \delta r\rangle)$$

Note first that $g_* \delta y = (A\delta r, A\delta u)$. Hence, defining $\sigma := \mathrm{d}\omega$:

$$\begin{aligned} g^*\sigma(\delta y, \delta' y) &= \sigma(g_*\delta y, g_*\delta' y) = k(\langle A\delta u, A\delta' r\rangle - \langle A\delta' u, A\delta r\rangle) \\ &= k(\langle \delta u, \delta' r\rangle - \langle \delta' u, \delta r\rangle) \quad \text{(By orthogonality)} \\ &= \sigma(\delta y, \delta' y) \end{aligned} \tag{1.1}$$

Hence this action preserves ω, so it is Hamiltonian. This then tells us that the moment map exists.

(c). We know that the Lie Algebra of the Euclidean Group consists of tuples of the form $Z = (j(\alpha), \gamma) \in \mathbb{R}^6$, where $j(\alpha)$ acts on an element by taking the cross product. We identity the dual $\mathfrak{g}^*$ by elements of the form $x = (l, p) \in \mathbb{R}^6$. We can define our action:

$$\langle x, Z\rangle := \langle l, \alpha\rangle + \langle p, \gamma\rangle$$

Then we can find our coadjoint action by G:

(1.2)

$$
\begin{aligned}
\left\langle \begin{bmatrix} I & c \\ 0 & 1 \end{bmatrix} \begin{bmatrix} l \\ p \end{bmatrix}, \begin{bmatrix} j(\alpha) & \gamma \\ 0 & 0 \end{bmatrix} \right\rangle &= \left\langle \begin{bmatrix} l \\ p \end{bmatrix}, \begin{bmatrix} I & -c \\ 0 & 1 \end{bmatrix} \begin{bmatrix} j(\alpha) & \gamma \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & c \\ 0 & 1 \end{bmatrix} \right\rangle \\
&= \left\langle \begin{bmatrix} l \\ p \end{bmatrix}, \begin{bmatrix} j(\alpha) & \alpha \times c + \gamma \\ 0 & 0 \end{bmatrix} \right\rangle \\
&= \langle l, \alpha \rangle + \langle p, \alpha \times c + \gamma \rangle \\
&= \langle l, \alpha \rangle + \langle c \times p, \alpha \rangle + \langle p, \gamma \rangle \quad \text{(triple product)} \\
&= \langle l + c \times p, \alpha \rangle + \langle p, \gamma \rangle
\end{aligned}
$$

Hence, the coadjoint action takes $(A, c) \cdot (l, p) = (Al + c \times Ap, Ap)$. This allows us to compute the coadjoint orbit immediately as just TS^2, since we can consider the action on $x_0 = (0, e_3) \mapsto (c \times Ae_3, Ae_3)$. If we are given arbitrary $(r, u) \in TS^2$, we first take u and complete as above to an orthonormal basis, and set $A = (u_1, u_2, u)$. Then, to get r, we just solve $(c \times u = r$, so we can take $c = u \times r$. Then pair is such that $(A, c) \cdot (0, e_3) = (r, u) \in TS^2$.

In order to find our moment map, we need $\Phi : X \to \mathfrak{g}^*$ such that

$$
\mathrm{d}\langle \Phi(\cdot), Z \rangle = -i_Z \sigma
$$

Where σ is the 2-form computed in part (b). The infinitesimal action of Z can be computed as sending $(r, u) \mapsto (\alpha \times r + \gamma, \alpha \times u)$. Using this:

$$
\begin{aligned}
i_Z(\delta y, Z(y)) &= k(\langle \delta u, \alpha \times r + \gamma \rangle - \langle \delta r, \alpha \times u \rangle) \\
&= \langle r \times \delta(ku), \alpha \rangle + \langle \delta(ku), \gamma \rangle - \langle ku \times \delta r, \alpha \rangle \\
&= \delta(\langle r \times (ku), \alpha \rangle + \langle ku, \gamma \rangle) \\
&= \langle \Phi(y), Z \rangle
\end{aligned} \tag{1.3}
$$

Where the above tells us that we should define our moment map as

$$\Phi(r,u) = \big(r \times (ku), ku\big)$$

2. Problem 2

(a). Rewriting our vector fields:

$$V(x,y,z) = x\partial_x - 2y\partial_y$$
$$W(x,y,z) = xy\partial_y - xz\partial_z$$

Then, we first calculate:

$$V(x,y,z)(xy\partial_y) = xy\partial_y + x^2y\partial_{xy} - 2xy\partial_y - 2xy^2\partial_{yy}$$
$$V(x,y,z)(-xz\partial_z) = -xz\partial_z - x^2z\partial_{xz} + 2xyz\partial_{yz}$$
$$W(x,y,z)(x\partial_x) = x^2y\partial_{xy} - x^2z\partial_{xz}$$
$$W(x,y,z)(-2y\partial_y) = -2xy\partial_y - 2xy^2\partial_{yy} + 2xyz\partial_{zy}$$

Now, subtract the last two terms from the top two, and this yields our Lie Bracket $[V,W](x,y,z)$. Doing this:

$$[V,W](x,y,z) = xy\partial_y - xz\partial_z = W(x,y,z)$$

And hence this is an involutive distribution. Using this, we know that there exists an integral manifold for this distribution. Computing our flows:

$$e^{tV}(x,y,z) = (xe^t, ye^{-2t}, z)$$
$$e^{tW}(x,y,z) = (x, ye^{xt}, ze^{-xt})$$

Composing these flows for time t and s, we find that the integral manifold is spanned by all curves of the form:

$$z = \frac{C}{x^2 y}$$

For $x, y, z > 0$. Hence, this solves the problem (the computations for this are not hard, just compose and then solve for s, t in terms of the coordinates x, y, and substitue these values for the z coordinate).

(b). The orthogonal complement to $V := x\partial_x + xy\partial_y + z\partial_z$ can be computed by selecting constants a, b, c such that $ax + bxy + cz = 0$.

Setting $a = 0$, we find $(0, -z, xy) \in V^{\perp}$. Setting $c = 0$, we also find $(-y, 1, 0) \in V^{\perp}$. Computing the bracket gives merely $(0, 0, -1) \notin V^{\perp}$, so this distribution is not involutive and hence not integrable (Frobenius).