NOTES IN COMMUTATIVE ALGEBRA: PART 2

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1. Completion of a Ring/Module

Here we shall consider two seemingly different constructions for the completion of a module and show that indeed they are isomorphic. Many standard results and definitions from topology shall be employed and assumed familiar here. We first have:

Definition 1.1. Let M be a group endowed with some topology. Then M is called a topological group if the mappings $M \times M \to M$, $(m, n) \mapsto mn$ and $M \to M$, $m \mapsto m^{-1}$ are continuous with respect to this topology.

Indeed for the above case, we immediately see that inversion and multiplication are in fact homeomorphisms.

Definition 1.2. Consider an A-module M endowed with some topology. M is called a *topological module* if M is a topological group with respect to addition, and scalar multiplication $A \times M \to M$, $(a, m) \mapsto am$ is continuous when A is endowed with the discrete topology.

Here we shall be interested in the case when we are actually dealing with a neighborhood filter consisting of submodules. We have the following:

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Lemma 1.3. *Let M be an A-module with $\mathcal{F} = \{M_\lambda\}_{\lambda\in\Lambda}$ a neighborhood filter of submodules (Λ is a directed set). Then, $\mathcal{B} := \{x + M_\lambda : \ x \in M, \ \lambda \in \Lambda\}$ is a basis for a topology on M.*

Under the assumptions of the above lemma, we shall refer to the neighborhood filter $\{M_\lambda\}_{\lambda\in\Lambda}$ as merely a filter of submodules of M. Intuitively, this tells us that a filter of submodules immediately induces a topology on any module, which we will call the *linear topology* induced by this filter, and the basis consists of all translations by some $x \in M$. We can now talk about convergence in terms of this filter.

Definition 1.4. Let M be a topological module with a filter of submodules. We say that a net x_λ converges to $x \in M$ if for every λ there exists a submodule M_μ such that $x_\gamma - x \in M_\mu$ for all $\gamma \geqslant \lambda$.

Cauchy sequences are defined in the obvious fashion with respect to the above definition. Let us suppose for now that the topology induced in Hausdorff, in which we know that limits will be unique. Then, we can take some motivation from elementary analysis. In the real number system, it is well known that every Cauchy sequence is in fact convergent to a real number.

Then we want to consider the set of all Cauchy sequences C on an A-module M with topology induced by the submodule filter $\{M_\lambda\}_{\lambda\in\Lambda}$. It is obvious that this set can be given an A-module structure by defining the natural addition and action of A as $\{x_n\} + \{y_n\} := \{x_n + y_n\}$, $a\{x_n\} := \{ax_n\}$.

We now consider the set of all sequences converging to 0, which we shall denote C_0. This in fact encompasses all convergent sequences with respect to the above topology since if $x_n \to x$, $x_n - x \to 0$. It is not so difficult to see that this is a submodule of C, so that we can talk about taking quotients.

Consider the quotient module C/C_0. Then, we define this to be the completion of M with respect to the topology induced by our submodule filter. There is a natural module homomorphism $\theta : M \to \widehat{M} := C/C_0$ taking $m \mapsto \{m\} + C_0$, $\{m\}$ just being the constant sequence. If it is the case that θ is an isomorphism, then we say that M is complete with respect to the induced topology. Since C_0 is a submodule, we get the immediate result:

Proposition 1.5. *Every field F is complete.*

Proof. Viewing F as a module over itself, just note that the kernel of our map θ is an ideal and hence equals either 0 or F. The only possible submodule filter inducing a Hausdorff topology would consist of $\{0\}$ after some truncation. If the kernel is F, then $F = 0$ identically since this would imply that every constant sequence converges to 0. Hence, $\operatorname{Ker}\theta = \{0\}$ and by the first isomorphism theorem, $F \cong C/C_0$ and F is complete by definition. □

Although the above construction of $\widehat{M}$ may be more intuitive, it can be unwieldy in practice. Let us consider another construction that will yield something a little more concrete.

Let $\{M_\lambda\}_{\lambda \in \Lambda}$ be a submodule filter for an A-module M. Then, we can of course take this submodule filter as a base of neighborhoods at 0. Given any M_λ, we see that M_λ^c can be written as a union of cosets and is hence open. But M_λ itself is open, so it must be both open and closed. Then we automatically find that the quotient topology on M/M_λ is in fact discrete.

Now consider the family of mappings $\phi_{\lambda\mu} : M/M_\mu \to M/M_\lambda$ where $m + M_\mu \mapsto m + M_\lambda$. We can then construct the inverse system $\{M_\lambda;\ \phi_{\lambda\mu}\}$. We can now consider the following set:

$$\varprojlim M/M_\lambda := \{\vec{m} \in \prod_\lambda M/M_\lambda :\ m_\lambda = \phi_{\lambda\mu}(m_\mu) \text{ for all } \lambda \leqslant \mu\}$$

This set $\varprojlim M/M_\lambda$ is called the inverse limit. We define this as the completion of our module. Intuitively, the inverse limit glues these sets together in terms of the associated mappings.

Now, give each M/M_λ the discrete topology and $\prod M/M_\lambda$ the product topology. Then, $\widehat{M} \subset \prod M/M_\lambda$ inherits the subspace topology and we can consider the natural map $\psi : M \to \widehat{M}$ defined by sending $m \mapsto \prod(m + M_\lambda)$. This map is continuous as the composition with the projection $p_\lambda : \widehat{M} \to M/M_\lambda$ is continuous, and $\widehat{M}/\operatorname{Ker} p_\lambda \cong M/M_\lambda$ since the projection is clearly surjective.

It can be shown that the linear topology defined by taking $\operatorname{Ker} p_\lambda$ as our submodule filter on $\widehat{M}$ coincides with the previously defined topology on $\widehat{M}$. This means that performing the same construction on $\widehat{M}$ yields a module isomorphic and indeed homeomorphic to $\widehat{M}$.

Whenever the map $\psi : M \to \widehat{M}$ is an isomorphism, we say that M is complete. The previous paragraph shows that the completion is complete, as expected.

Now it is natural to ask how this is related to the previous construction. The following theorem will clear this up:

Theorem 1.6. *With $\widehat{M} = C/C_0$ are defined before, we have an isomorphism $\alpha : \varprojlim M/M_\lambda \to \widehat{M}$ defined by taking $(m_\lambda + M_\lambda)_{\lambda \in \Lambda} \mapsto \{m_\lambda\}_{\lambda \in \Lambda} + C_0$.*

Furthermore, with

$$\begin{aligned}\theta :& M \to \widehat{M} \\ & m \mapsto \{m\} + C_0\end{aligned}$$

And

$$\begin{aligned}\eta :& M \to \varprojlim M/M_\lambda \\ & m \mapsto (m + M_\lambda)_{\lambda \in \Lambda}\end{aligned}$$

The following diagram commutes:

$$\begin{tikzcd} \varprojlim M/M_\lambda \arrow[r, "\alpha"] & \widehat{M} \\ & M \arrow[ul, "\eta"] \arrow[u, "\theta"'] \end{tikzcd}$$

With this, we can now consider the most natural cases. For an A-module M, there is a natural filter $IM \supset I^2M \supset I^3M \supset I^4M \supset \dots$. Then, the completion with respect to this filter is called the I-adic completion and is denoted $\widehat{M}_I$. Of particular interest is when I is a maximal ideal $\mathfrak{m}$. In this case we have an analog to the localization of a ring with respect to some prime ideal $\mathfrak{p}$.

Proposition 1.7. *The completion of a ring R with respect to a maximal ideal is a local ring with maximal ideal*

$$\widehat{\mathfrak{m}}_{\mathfrak{m}} = \{(0, a_2+\mathfrak{m}^2, a_3+\mathfrak{m}^3, \dots) \in \prod R/\mathfrak{m}^i : \ a_i \equiv a_j \mod \mathfrak{m}^i \text{ for all } j > i\}$$

Example 1.8. Consider the ring of polynomials $R := k[x_1, \dots, x_n]$ over a field k. Then, $(x_1, \dots, x_n) := \mathfrak{m}$ is a maximal ideal and the

$\mathfrak{m}$-adic completion can be viewed as the ring of formal power series $k[[x_1, \ldots, x_n]]$.

To see this, define $\phi : \widehat{R}_{\mathfrak{m}} \to k[[x_1, \ldots, \mathbf{x}_n]]$ by sending $f \mapsto (f + \mathfrak{m}, f + \mathfrak{m}^2, \ldots)$. The preimage of any $(f_1 + \mathfrak{m}, f_2 + \mathfrak{m}^2, \ldots)$ can be computed as $f_1 + (f_2 - f_1) + (f_3 - f_2) \ldots$, and this is trivially a homomorphism. Hence these rings are isomorphic.

Example 1.9. Consider the ring $\mathbb{Z}$. Then, consider the completion with respect to any prime ideal (p), where p is a prime number. This is also maximal, and we can view $\mathbb{Z}$ as the ring $\mathbb{Z}[p]$ by writing any integer in its base p expansion. Then the completion of $\mathbb{Z}$ with respect to this maximal ideal is of the form $\mathbb{Z}[[p]]$, where our addition is defined in base p.

This is known as the ring of p-adic integers and is denoted $\mathbb{Z}_p$. These numbers are often written merely as digits of the form $\ldots a_n \ldots a_3 a_2 a_1 a_0$, where $0 \leqslant a_i < p$ is the coefficient of p^i in our power series expansion, and have some interesting properties.

For example, in $\mathbb{Z}_2$, $\ldots 1111 + 1 = 0$, so that $\ldots 1111 = -1$. Expanding this out in terms of the power series definition, we find that $1 + 2 + 4 + 8 + \cdots = -1$ in the ring of 2-adic integers.

Indeed, in general we see that $(p-1)(1 + p + p^2 + \ldots) = -1$ in $\mathbb{Z}_p$.