HOMEWORK 2 COMPLEX ANALYSIS

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1. Problem 1

(1) For $|z - z_1| = |z - z_2|$, merely recognize that this is the set of all $z \in \mathbb{C}$ that are equidistant to z_1 and z_2. This is geometrically represented as the perpendicular bisector of the line segment connecting z_1 to z_2.

(2) If $1/z = \bar{z}$, then $1 = z\bar{z} = |z|^2$. Thus this set is just the unit circle in the complex plane.

(3) For $\text{Re}(z) = 3$, this is just a vertical line in the complex plane at the point $x = 3$, where $z = x + iy$.

(4) Similar to the previous set, $\text{Re}(z) > c$ would be the set of all $z \in \mathbb{C}$ lying strictly to the right of the line $x = c$ in the complex plane. Respectively for $\text{Re}(z) \geq c$ this is the set of all $z \in \mathbb{C}$ lying to the right of the line $x = c$, including the line itself.

(5) Let $a = a_1 + ia_2$ and $b = b_1 + ib_2$. Then, $\text{Re}(az + b) > 0 \implies a_1x - a_2y + b_1 > 0$, where $z = x + iy$. This then simplifies to $y < \frac{a_1}{a_2}x + \frac{b_1}{a_2}$. Since $\mathbb{R}^2 \cong \mathbb{C}$, this can merely be viewed in

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the xy plane as the set of points lying strictly below the line $y = \frac{a_1}{a_2}x + \frac{b_1}{a_2}$

(6) Let $z = x + iy$. Then, $|z| = \text{Re}(z) + 1$ becomes $x^2 + y^2 = x + 1$ which simplifies to $(x - \frac{1}{2})^2 + y^2 = \frac{5}{4}$, which, by the same reasoning as in (5), can be viewed as the circle centered at $(\frac{1}{2}, 0)$ with radius $\frac{\sqrt{5}}{2}$ in the xy plane.

(7) $\text{Im}(z) = c$ can be viewed as a horizontal line at height c in the xy plane.

2. Exercise 3

Proof. Given $\omega = se^{i\phi}$, by Euler's formula (I should probably clarify, the one that says $e^{i\phi} = \cos(\phi) + i\sin(\phi)$) we know that $e^{i\phi}$ is periodic of period 2π. Thus we really have the following:

$$\omega = se^{i(\phi+2\pi k)}$$

For any $k \in \mathbb{Z}$. Then, solving $z^n = \omega$ leads to

$$z = s^{1/n}e^{i(\phi+2k\pi)/n}$$

Which gives distinct values for $k = 0 \ldots n-1$, implying that there are precisely n distinct solutions to this equation. $\square$

3. Exercise 12

Proof. Define $f(z) := \sqrt{|x||y|}$. Then, clearly $\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$. The Cauchy-Riemann equations yield the following:

$$\sqrt{\frac{|y|}{|x|}} = 0$$

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Implying f is differentiable at $x = y = 0$. However, since the Cauchy-Riemann equations hold for *no* deleted neighborhood of the origin, f is holomorphic nowhere. □

4. Exercise 13

Proof. First suppose that $f(z) = u + iv$ is holomorphic and that $\text{Re}(f)$ is a constant. Then, $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$. Employing the Cauchy-Riemann equations yields:

$$\frac{\partial v}{\partial x} = 0$$
$$\frac{\partial v}{\partial y} = 0$$

The condition $\frac{\partial v}{\partial x} = 0$ implies $v = v(y)$, which then means $\frac{\partial v}{\partial y} = \frac{dv}{dy} = 0$, from which we can conclude that v is a constant. Thus, u and v are both constants, so f itself is constant. □

Actually, for part 2 of this problem, you can literally copy and paste this entire argument and interchange the order of u and v to find the same conclusion... which is precisely what I am going to do.

Proof. Second suppose that $f(z) = u + iv$ is holomorphic and that $\text{Im}(f)$ is a constant. Then, $\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$. Employing the Cauchy-Riemann equations yields:

$$\frac{\partial u}{\partial x} = 0$$
$$\frac{\partial u}{\partial y} = 0$$

The condition $\frac{\partial u}{\partial x} = 0$ implies $u = u(y)$ which then means $\frac{\partial u}{\partial y} = \frac{du}{dy} = 0$, from which we can conclude that u is a constant. Thus, u and v are both constants, so f itself is constant. □

Proof. Lastly suppose that we have $|f| = u^2+v^2$ is constant everywhere (where $f = u + iv$). Then

$$\frac{\partial |f|}{\partial z} = 0 = \frac{1}{2}\Big(\frac{\partial(u^2+v^2)}{\partial x} + \frac{1}{i}\frac{\partial(u^2+v^2)}{\partial y}\Big)$$

After some simplification we find that:

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} - i\Big(u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y}\Big) = 0$$

After equating real and imaginary parts we find the following system:

$$\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = 0$$

Now the 2x2 matrix is merely the Jacobian J_f of f. Then, if $\mathrm{Det}(J_f) \neq 0$, J_f is invertible and we thus conclude that $u = v = 0$, so f is trivially constant. Thus assume u, $v \neq 0$. Then $\mathrm{Det}(J_f) = 0$. However, using the Cauchy Riemann equations, we have:

$$\mathrm{Det}(J_f) = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y} = \frac{\partial u}{\partial x}^2 + \frac{\partial v}{\partial x}^2 = 0$$

From which we conclude that all partial derivatives vanish for any $z \in \mathbb{C}$. Thus, f is constant, and the result is proved. $\square$